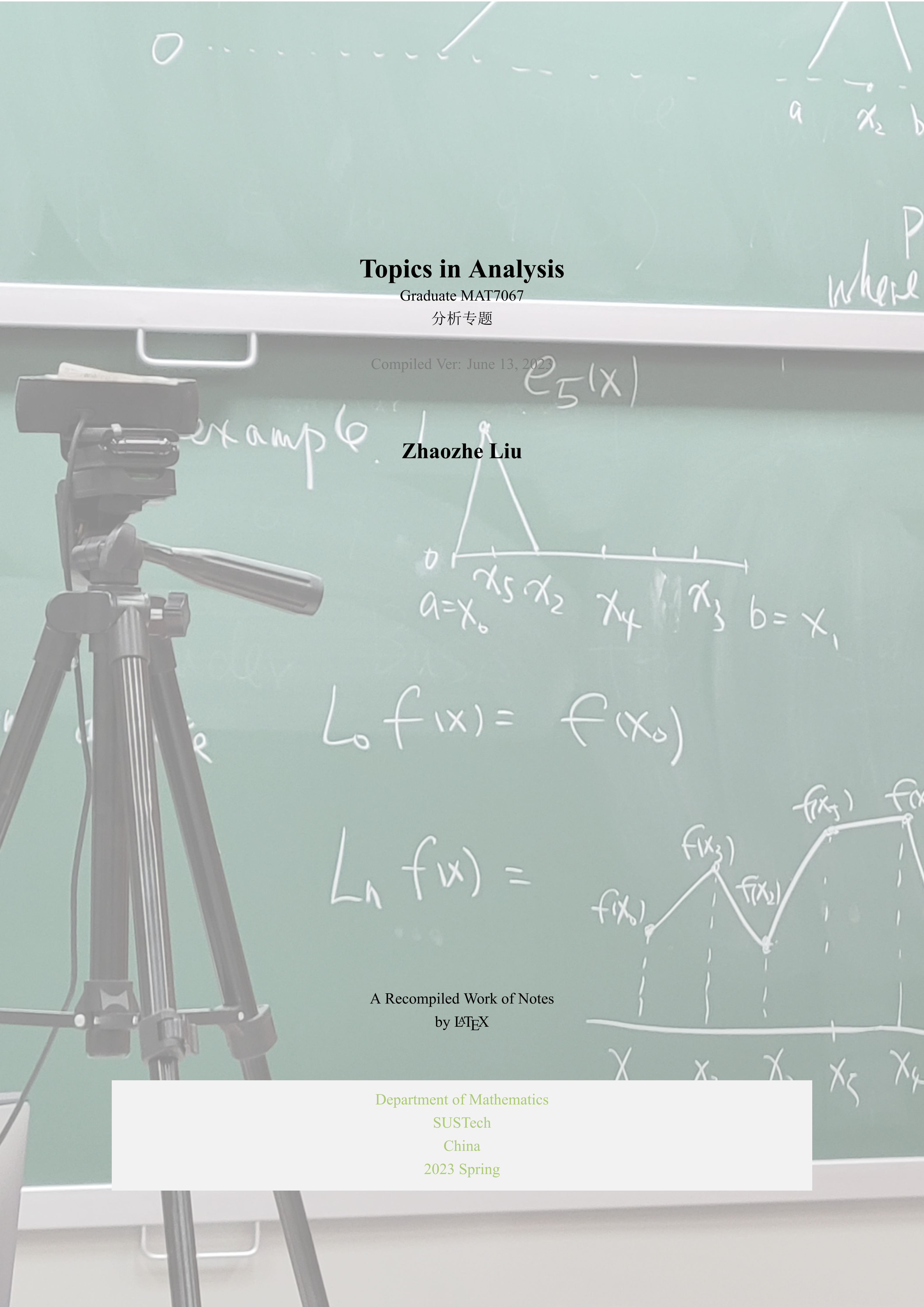


Topics in Analysis

Graduate MAT7067

分析专题

Compiled Ver: June 13, 2023

Zhaozhe Liu

A Recompiled Work of Notes

by LaTeX

Department of Mathematics

SUSTech

China

2023 Spring

Contents

课程信息

授课教师：刘博辰

Assistant Professor in College of Science, Department of Mathematics
Education：

- Ph.D in Mathematics, University of Rochester, USA, 2017
- B.S. in Mathematics (Shing-Tung Yau Honor Program), Zhejiang University, 2012

Professional Experience：

- 2020 - present Associate Professor, Department of Mathematics, Southern University of Science and Technology
- 2020 - 2020 Assistant Professor, Department of Mathematics, Southern University of Science and Technology
- 2019 - 2020 Postdoc, Mathematics Division, National Center for Theoretical Sciences, Taiwan
- 2018 - 2019 Postdoc, Department of Mathematics, Chinese University of Hong Kong
- 2017 - 2018 Postdoc, Department of Mathematics, Bar-Ilan University, Israel
- 2016 - 2017 Junior Research Assistant, Department of Mathematics, Chinese University of Hong Kong

Course Arrangement:

- 单周：周二、周四晚 7:00 点到 8:50
- 双周：周四晚 7:00 点到 8:50

Course Assessment:

- 期中考为三四个问题（No Return Paper）
- 期末考（教务系统登记为考察），大概率为 Take-Home Exam (Via E-Mail)
  6 月 13 日，下午三个小时

Part 1: An Introduction to Non-harmonic Fourier Series

Part 1 主要包含 4 个 Chapters：

- Chapter 1: Bases in Banach Spaces
- Chapter 2: Entire functions of exponential type
- Chapter 3: The completeness of sets of complex exponentials
- Chapter 4: Interpolation and Bases in Hilbert Space

The theory of **nonharmonic Fourier series** is concerned with the **completeness** and **expansion properties** of sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^p[-\pi, \pi]$.

Lecture 1: Chapter 1: Bases in Banach Spaces-Schauder basis, Schauder theorem and Orthonormal basis in Hilbert Spaces

Lecture 1-2023 年 2 月 14 日今晚有冷空气

主要内容：介绍了这门课程的主要目标：Generalization of fourier series on $[-\pi, \pi]$，以及 Chapter 1 的 Section 1.1, Section 1.2 和 Section 1.3 的部分内容。

- Example of generalization 1:
- Example of generalization 2: Extend Fourier series to more general ambient space such as $\mathbb{R}^d, \mathbb{Z}/p\mathbb{Z} \Rightarrow$ Fourier analysis on group, especially locally compact abelian group.

Theorem 1.1.1 (Schauder). *$C_{[a,b]}$ possesses a basis*

- Pay attention to the construction of $e_n(x)$, here we present an example of $e_5(x)$ illustrating how this construction is done.

Bernstein Polynomial

- ℓ^∞ has no basis

其他信息：Midterm 大约安排在 Part 1 结束，Part 2 开始之前

Main Goal: Generalization of Fourier series on $[-\pi, \pi]$, then

= when certain regularity

$$f(t) \sim \sum_{n\in\mathbb{Z}} a_n e^{int}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt.$$

Example of generalization ①: $f \sim \sum_{n=0}^{\infty} a_n e^{i\lambda_n t}$, find $\{\lambda_1, \cdots, \lambda_n, \cdots\}$

由 $n \in \mathbb{Z} \Rightarrow \lambda_n$, for such decomposition

Example of generalization ②: Extend Fourier series to more general ambient space.

e.g. on $[-\pi, \pi]$, $\mathbb{R}^d$, $\mathbb{Z}/p\mathbb{Z}$, etc. $\Rightarrow$ Fourier analysis on group.

We can generalize space mentioned abouve. Here we only consider locally compact Abelian group.

$\Downarrow$

We still have potential to go further. but that will not be covered in this course (Some French mathematicians)

Reference: Robert Young, and Folland's book.

First 1980 (Revised 2001) originates in Folland's lecture note in 1993.

Other reference: Rudin (Fourier analysis on group) 1960s : Hard to read! Assume solid background in Functional Analysis. Folland's book is more self-contained.

whose book is always harsh to read.

1960s – 1970s (before stein), when abstract harmonic flourished (by Rudin)

then Stein (more detailed style), then Bourgain, Wolff, ⋯, now.

Trailer: · Fourier inverse theorem $\Rightarrow$ Pontryagin duality, one of the few example We can see connection between category theory and analysis.

· Relative topic summer school (2023)

Assessment: · Midterm between Part 1 ~ part 2, only on the first part.
(TBA)

Within 8 weeks, maybe in-class exam, mainly from the reference book (at most one question from external source)

· Final. (may be on the part 2, TBA)

As is complicated.

Office Hour: Single Tuesday 4–6 p.m. (check e-mail)

Chapter 1: Bases in Banach spaces (only consider infinity-dim space as Finite-space is maily Linear algebra)

Let X be an infinite-dimensional Banach space over $\mathbb{C}$ or $\mathbb{R}$

Def: Hamel basis: maximal linearly independent subset (Existence support by the Zorn's lemma, Axiom of choice). but it's hard to actually find!

Def Schauder basis: $\{x_1, x_2, \cdots\} \subset X$ is a schauder basis for X, if every $x \in X$ corresponds to unique scalars $c_1, c_2, \cdots$. s.t. $x = \sum_{n=1}^{\infty} c_n x_n$, i.e.

$\lim_{n\to\infty} \| x - \sum_{i=1}^{n} c_i x_i \| = 0$

Remark: Although Hamel and Schauder basis are different in some way, Schauder basis is the default basis throughout part 1.

Remark: A Banach space with a basis must be separable

$\exists$ a countable dense subset.

proof: $\{\sum_{\text{finite}} c_i x_i,\ c_i \in \mathbb{Q} + i\mathbb{Q}\}$

e.g. ℓ^∞ has no basis

ℓ^p, $1 \le p < \infty$ has basis $\{(0, \cdots, 1, 0, \cdots)\}$

$\Downarrow$

Banach Asked in 1932: "Does every separable Banach space have a basis?"

Answered by Per Enflo in 1973: No (the counter-example is quite tedious. most familiar examples have basis"

$\Downarrow$

see exercise 1, 3, 6, 7, P2

Section 1.2: Schauder basis for $C[a,b]$

continuous functions on $[a,b]$ with norm $\|f\| = \max_{a \le t \le b} |f(t)|$

Recall the Weierstrass approximation theorem: $\forall \varepsilon > 0$, $\forall f \in C[a,b]$, $\exists$ polynomial P s.t. $\|f - P\| < \varepsilon$

- possible approaches (there are many different ways)

Bernstein polynomial, $n = 0, 1, 2, \cdots$

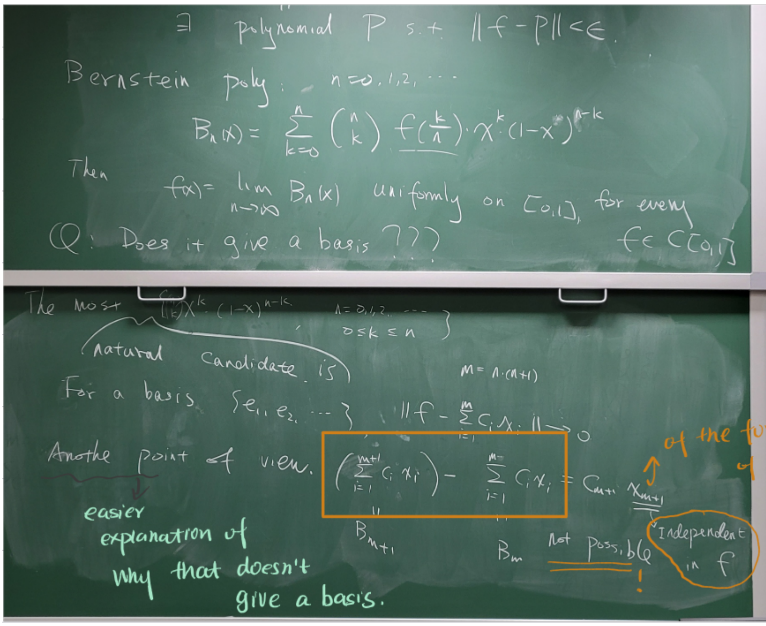
$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

then $f(x) = \lim_{n\to\infty} B_n(x)$ uniformly on $[0,1]$ for every $f \in C[0,1]$

that leads to Q: Does it give a basis? : most natural candidates $\{x^k (1-x)^{n-k}, n = 0, 1, \cdots\}_{0 \le k \le n}$

but there will be problem in convergence, as we require $\|f - \sum_{i=1}^{m} c_i \overset{\text{basis}}{\uparrow}{x_i}\| \to 0$ (only convergence for a sub-sequence)

Another point of view:

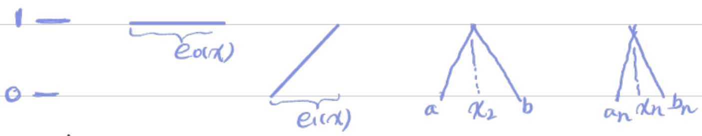


of the form of a basis $\{x^k(1-x)^{n-k}\}$

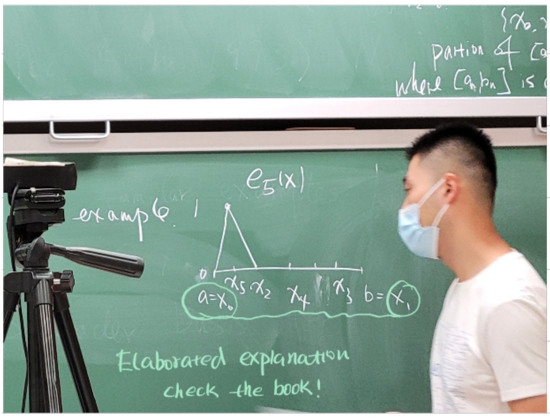
Theorem (Schauder): $C[a,b]$ possess a basis.

Proof: let $\{x_0, x_1, \cdots\} \subset [a,b]$ be a countable dense subset, and $x_0 = a$, $x_1 = b$

and $e_0(x) = 1$, $e_1(x) = \frac{x-a}{b-a}$, $e_2(x)$, $\cdots$, $e_n(x)$

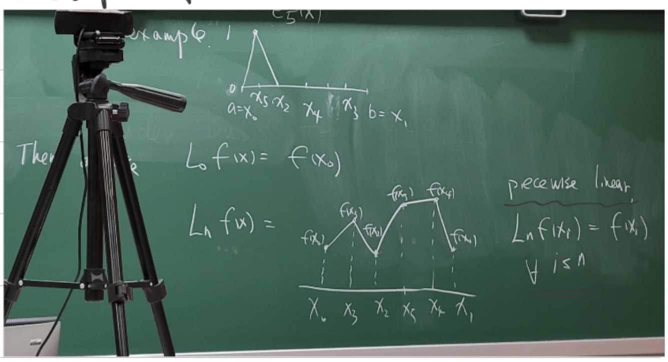


$\{x_0, x_1, \cdots, x_{n-1}\}$ gives a partition of $[a,b]$, and $x_n \in [a_n, b_n]$, where $[a_n, b_n]$ is an interval from this partition



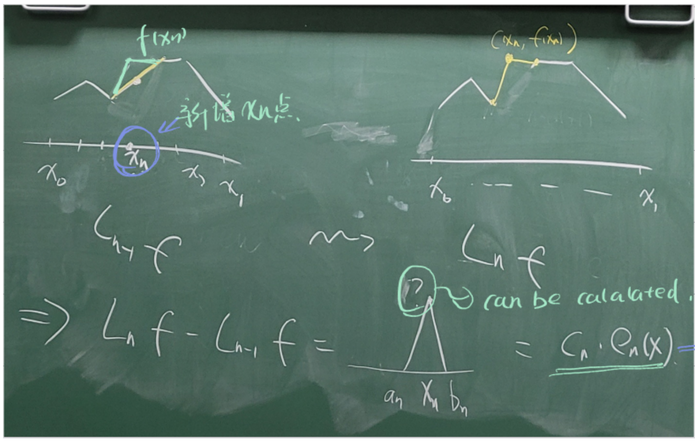
then dense $L_0 f(x) = f(x_0)$

$L_n f(x) =$



Clearly, $L_n f \to f$ uniformly on $[a,b]$, but how to connect e_n and $L_n(f)$?

First $f = \underset{f(x_0) = f(x_0)\cdot e_0(x)}{\underset{\|}{L_0 f}} + \sum_{n=1}^{\infty} (L_n f - L_{n-1} f)$ then what is $L_n f - L_{n-1} f$, see the following illustration



$\Rightarrow$ 可以写成以 $\{e_n\}$ 为基的形式

So what is C_n? In fact $\begin{cases} C_0 = f(x_0) \\ C_n = (f - L_{n-1} f)(x_n) \end{cases}$ $\Rightarrow$ Hence we have shown the convergence now, we may consider the uniqueness.

uniqueness: $f = \sum_{n=0}^{\infty} C_n e_n = \sum_{n=0}^{\infty} C_n' e_n$, then

$0 = \sum_{n=0}^{\infty} (C_n - C_n') e_n$, notice that $e_n(x_i) = 0$, for $i = 0, \cdots, n-1$

let $x = x_0 \Rightarrow C_0 = C_0'$, take $x = x_1 \Rightarrow C_1 = C_1'$

$x = x_2 \Rightarrow C_2 = C_2'$, $\cdots$ $\Rightarrow$ uniqueness.

$\blacksquare$

Exercise 3.4，后面我们主要考虑 Hilbert space.

Section 1.3 Orthonormal basis in Hilbert space.

somebook requires Hilbert space to be separable by def

In a separable Hilbert space $\mathcal{H}$. We say $\{e_1, e_2, \cdots\}$ is an othonormal basis, if it is a basis, and $(e_i, e_j) = \delta_{ij}$

inner product in this Hilbert space.

- An orthonormal basis $\Leftrightarrow$ A complete orthonormal sequence

$\operatorname{span}\{e_1, \cdots\}^{\perp} = \{0\}$

- Basis expansion $f = \sum \underbrace{(f, e_n)}_{\text{Fourier coefficients}} e_n$

- Parseval's Idenity $\|f\|^2 = \sum |(f, e_n)|^2$, more generally

$(f, g) = \sum (f, e_n) \cdot \overline{(g, e_n)}$

$\rightsquigarrow$ Conversely, if $f = \sum_{n=1}^{\infty} (f, e_n) e_n$, where $\{e_n\}$ is an orthonormal sequence then $\{e_n\}$ is a basis (It has the uniqueness)

Proof: It sufficies to show the uniqueness, if

$f = \sum C_n e_n = \sum C_n' e_n \Rightarrow 0 = \sum (C_n - C_n') e_n$, then

$0 = (0, e_n) = (\sum (C_n - C_n') e_n, e_n) = C_n - C_n' \Rightarrow C_n = C_n'$

Note that if we remove the orthogonality, then this result is False!

$f = \sum \langle f, e_n \rangle e_n \not\Rightarrow$ basis

Example: In $L^2[0,\pi] \subset \underset{\sim}{L^2[-\pi,\pi]}$

we have Fourier series. $f \sim \sum a_n e^{int}$, $a_n = \langle f, \underbrace{e^{int}}_{e_n} \rangle$

For $g \in L^2[0,\pi] \xrightarrow{\text{extend}} \tilde{g} \in L^2[-\pi,\pi]$ (补 0), then $g(t) = \sum \langle \tilde{g}, e_n \rangle_{L^2(-\pi,\pi)} e_n$ in $[0,\pi]$

$= \sum \langle g, e_n \rangle_{L^2[0,\pi]} e_n$ in $[0,\pi]$

That leads to $g = \sum \langle g, e_n \rangle e_n$, $\forall g \in L^2[0,\pi]$, the uniqueness fails, as the extension from $L^2[0,\pi] \to L^2[-\pi,\pi]$ is not unique. (不一定补 0 的 natural extension)

这时 $\{e_n\}$ 在 $[-\pi,\pi]$ 上正交，但在 $[0,\pi]$ 上积分少了一半，就不正交了！

Trailer: In $\frac{1}{3}$ of this class, we will consider some scenarios like this (expansion not unique)

$\Downarrow$

Named Fourier frame. Also quite useful.

补充笔记

第一篇

Separable space: A topological space is called **separable** if it contains a countable, dense subset. To be explicit, X is separable if there exists an infinite sequence $a : \mathbb{N} \to X$ such that, given any point b in X and any neighbourhood U of b, we have $a_i \in U$ for some i.

可分性的意义在于: 在一个可数稠子集上往往较容易获得某些所期待的结论, 而这种结论有可能**通过一个极限过程过渡到全空间**。可以说，在某种意义上可分空间“比较小”, 正如通常认为有可数基的拓扑“比较小”一样。实际上，**可分性与第二可数性确有很强的联系**。[1]

Definition 1.3.1. 第二可数空间 (second countable space), 或满足第二可数性公理的空间, 即具有可数拓扑基的拓扑空间。

Proposition 1.3.2. 命题：第二可数的拓扑空间是可分的. 可分度量空间是第二可数的.

Per Enflo (1944-), a Swedish mathematician working primarily in functional analysis.

[1] 参考「胡适耕」抽象空间引论 P.85

Lecture 2: cont.Hilbert Space, Reproducing Kernel

Lecture 2-2023 年 2 月 16 日天气晴好 ☼，13°C-21°C

主要内容：Functional Hilbert Space, H^2 Hardy space, A^2 space, Paley-Wiener Space

Section 1.4: Reproducing kernel
A^2 Space 在 Riemann 几何中也有使用

其他信息：

cont. Hilbert Space.

Parseval's identity, $\|f\|^2 = \sum |\langle f, e_n\rangle|^2$

more generally $\langle f, g\rangle = \sum \langle f, e_n\rangle \overline{\langle g, e_n\rangle}$

this gives an isomorphism between $\mathcal{H}$ (Hilbert space) and ℓ^2, in particular, separable Hilbert spaces are all isomorphism.

Bessel inequality: if $\{e_1, e_2, \cdots\}$, an orthonormal sequence, then

$\sum |\langle f, e_n\rangle|^2 \le \|f\|^2$

e.g. ① ℓ^2 with natural basis

or $L^2[0,1]$ in other scenes

② Fourier series $L^2[-\pi,\pi]$, with $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f \cdot \bar{g}$, with orthonormal basis $e_n(t) = e^{int}$

to see this first orthogonality ✓, It remaing to prove the completeness i.e. $\langle f, e_n\rangle = 0\ \forall n \Rightarrow f = 0$

It's easy to see when f is continuous.

$\|f\|^2 = \langle f, f\rangle = \langle f - P_n, f\rangle$ ($\forall$ trigonometric series)

$\le \max_{[-\pi,\pi]} |f - P_n| \cdot \int_{-\pi}^{\pi} f$

$\le c \cdot \|f\| \cdot O(1)_{n\to\infty}$ $\Rightarrow \|f\| = 0$

For general $f \in L^2[-\pi,\pi]$, let $g(t) = \int_{-\pi}^{t} f(x)\,dx$ ($g(t)$ is continuous! by f being integrable)

Notice $\langle g, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cdot e^{-int}\,dt$

Integration by parts

$= \frac{-1}{2\pi i n}\left[g(t) e^{-int}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} g'(t) e^{-int}\,dt \right]$ ($g' = f$ a.e)

$= 0$, as $g(-\pi) = g(\pi) = 0$, and $\langle f, e_n\rangle = 0$

$\Rightarrow \langle g - \langle g, e_0\rangle e_0, e_n\rangle = 0, \forall n \Rightarrow g$ is a constant

$\Rightarrow f = g' = 0$ a.e. ▥

As a consequence, $f(t) = \sum \underbrace{\langle f, e_n\rangle e_n}_{\text{unique}} \overset{\text{def}}{=} \sum \hat{f}(n) e^{int}$

and $\|f\|^2 = \sum |\hat{f}(n)|^2$

$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2$

Remark: Carleson (1966) point-wise convergence a.e.

proposition: $\sum_{n\in\mathbb{Z}} |\hat{f}(n+A)|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$, $\forall A \in \mathbb{R}$.

proof: by definition $\hat{f}(n+A) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{-i(n+A)t}\,dt$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t) e^{-iAt}] e^{-int}\,dt$

$\Rightarrow \sum |\hat{f}(n+A)|^2 = \|f(t)\cdot e^{-iAt}\|^2 = \|f\|^2 = \sum |\hat{f}(n)|^2$ ▥

Example: take $f=1$, then $\hat{f}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ixt}\,dt=\frac{\sin\pi x}{\pi x}$

Now let $A=\frac{t}{\pi}$, then

$$\sum\left[\frac{\sin(n\pi+t)}{n\pi+t}\right]^2=\|f\|^2=1$$

$$\Rightarrow \frac{1}{\sin^2(t)}=\sum_{n=-\infty}^{+\infty}\frac{1}{(n\pi+t)^2},\ \forall t\neq 0.$$

cont. eg. ③ The Hardy space $H^2=\left\{\begin{array}{l}\text{Analytic functions } f=\sum C_n z^n \\ \text{in } \mathbb{D}, \text{ with } \sum|C_n|^2<\infty\end{array}\right\}$ 开圆盘

with $(f,g)=\sum a_n\overline{b_n}$, if $f=\sum a_n z^n$, $g=\sum b_n z^n$

H^2 is a Hilbert space and z^n as orthonormal basis. In fact H^2 is a closed subspace of $L^2[-\pi,\pi]$, $\ell^2\triangleq H^2$ (Hilbert space isomorphism)

As $\sum_{n=0}^{\infty}C_n z^n\longmapsto\sum_{n=0}^{\infty}C_n e^{int}$

the following example is more interesting (unit disk)

④ $A^2=\left\{\begin{array}{l}\text{analytic function } f \text{ in } \mathbb{D} \\ \text{with } \iint_{|z|<1}|f(z)|^2dxdy<\infty\end{array}\right\}$

先求和再积分

先积分再求和.

Question: connection between A^2 and $\mathcal{H}^2$: $\mathcal{H}^2\subseteq A^2$ (A^2的 norm 比 $\mathcal{H}^2$小)

A non-trivial inclusion!

Recall the most common example of divergent sequence $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}$, then consider

$f(z)=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}z^n\notin\mathcal{H}^2$ $\because\sum_{n=0}^{\infty}\frac{1}{(n+1)}=\infty$

then $\|f\|_{A^2}^2=\iint_{|z|<1}\left|\sum_{n=0}^{\infty}\overset{C_n}{\frac{1}{\sqrt{n+1}}}z^n\right|^2dxdy$, since integral on unit disc, use the polar coordinate

$$=\int_0^1\int_0^{2\pi}\underbrace{\left|\sum_{n=0}^{\infty}\frac{r^ne^{in\theta}}{\sqrt{n+1}}\right|^2}_{\text{Fourier series}}d\theta\cdot r\,dr$$

$\forall r<1$

$$=\int_0^1 2\pi\cdot\sum_{n=0}^{\infty}\frac{r^{2n+1}}{n+1}dr=\sum_{n=0}^{\infty}\frac{\pi}{(n+1)^2}<\infty.$$

More generally, $\forall f=\sum_{n=0}^{\infty}c_nz^n$, $\|f\|_{A^2}^2=\pi\cdot\sum_{n=0}^{\infty}\frac{|C_n|^2}{n+1}$.

Question: Find orthonormal basis of A^2

$$=\sum_{n=0}^{\infty}a_ne_n\longrightarrow\sum_{n=0}^{\infty}|a_n|^2$$

then $\boxed{e_n=\sqrt{\frac{n+1}{\pi}}z^n}$ √

and $(f,g)_{A^2}=\sum_{n=0}^{\infty}\frac{a_n\overline{b_n}}{n+1}$, $f=\sum_{n=0}^{\infty}a_nz^n$, $g=\sum_{n=0}^{\infty}b_nz^n$

Ref Exercise 16, 以上的另一种证法, 这节有17个 exercises

Remark: From $\|f\|_{A^2}^2=\pi\cdot\sum_{n=0}^{\infty}\frac{|C_n|^2}{n+1}$, it is easy to see A^2 is complete, thus a Hilbert space, an alternative proof check Ex. 16

check Exercise 2, 4, 12, 13, 16, 17

important tool, rather abstract · 可以把 f 表达出来 hence the name.

Section 1.4: Reproducing Kernel.

Def : (functional Hilbert space) Hilbert space of functions

Let S be a set, and $\mathcal{H}$ be a Hilbert space whose elements are functions on S. We say $\mathcal{H}$ is a functional Hilbert space if $\forall x \in S$, the map $f \mapsto f(x)$ is bounded from $\mathcal{H}$ to $\mathbb{C}$ evaluation

That is $\forall x \in S$, $\exists M_x > 0$ s.t.

$$|f(x)| \le M_x \|f\|, \quad \forall f \in \mathcal{H}$$

e.g. ① $\ell^2 = \{\text{functions } f \text{ on } \mathbb{Z},\ \sum_{n=-\infty}^{+\infty} |f(n)|^2 < \infty\}$

$$\forall n \in \mathbb{Z},\ |f(n)| \le \|f\| = \left(\sum |f(n)|^2\right)^{\frac{1}{2}} < \infty$$

$\Rightarrow \ell^2$ is a functional Hilbert space ($M_x = 1$, $\forall x \in \mathbb{Z}$)

② $L^2[-\pi,\pi]$ is $\boxed{\text{not}}$ a functional Hilbert space

i.e. evaluation not well-define

($f \mapsto f(x)$ not well-defined, rigor proof later)

Now let $\mathcal{H}$ be a functional Hilbert space i.e. $\forall x \in S$.

$f \mapsto f(x)$ is a bounded linear functional on $\mathcal{H}$

By Riesz representation theorem : $\exists K_x \in \mathcal{H}$ s.t. $f(x) = \langle f, K_x \rangle$

Notice $K_y(x) = \langle K_y, K_x \rangle \overset{\text{def}}{=} K(x,y)$, reproducing kernel of $\mathcal{H}$ or kernel function

$$\overline{K_x(y)} = \overline{\langle K_x, K_y \rangle}$$

Proposition: If $\{e_1, \cdots\}$ is an orthonormal basis, then

$$\boxed{K(x,y) = \sum e_n(x)\overline{e_n(y)}}$$

def of reproducing kernel in some references

Proof: $K(x,y) = K_y(x) = \sum \langle K_y, e_n \rangle e_n(x) = \sum \overline{e_n(y)}\, e_n(x)$ ▥

Corollary: $L^2[-\pi,\pi]$ is not a functional Hilbert space

Proof: orthonormal basis e^{int}, $K(t,t) = \sum |e^{int}|^2 = \infty$ ▥

Actually $\mathcal{H}^2$, A^2 are both functional Hilbert space $L^2 \xrightarrow{\text{mod}}$ [illegible] $\boxed{\text{paley-weiner space}}$ ✓ Important later, mentioned in stein's complex analysis

Example: $\mathcal{H}^2$, $\forall f(z) = \sum c_n z^n \in \mathcal{H}^2$, $\forall \beta \in \mathbb{D}$

$$|f(\beta)| = \left|\sum c_n \beta^n\right| \overset{\text{Hölder}}{\le} \underbrace{\left(\sum |c_n|^2\right)^{\frac{1}{2}}}_{= \|f\|_{\mathcal{H}^2}} \cdot \underbrace{\left(\sum |\beta|^{2n}\right)^{\frac{1}{2}}}_{M_\beta}$$

$|f(\beta)| \le M_\beta \|f\|_{\mathcal{H}^2}$

$\Rightarrow \mathcal{H}^2$ is a functional Hilbert space, and $\overset{\sum c_n z^n}{f(z)} = \langle f, K_z \rangle = \sum c_n \overline{a_n}$, if $K_z(w) = \sum a_n w^n$

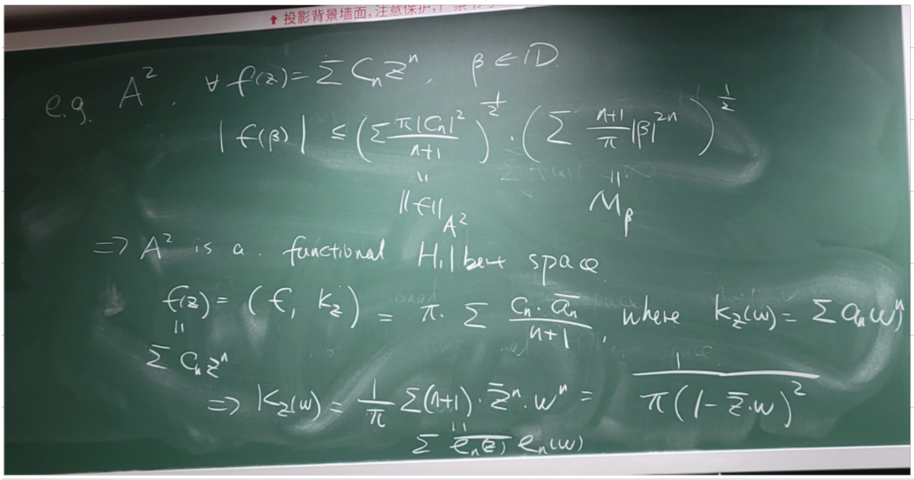
$$\Rightarrow k_z(w) = \sum (\bar{z})^n w^n = \frac{1}{1-\bar{z}w} \quad \text{(Szegő kernel)}$$
$$\sum e_n(w)\overline{e_n(z)}$$

$\blacksquare$

Example: A^2, $\forall f(z) = \sum c_n z^n$, $\beta \in \mathbb{D}$

$$|f(\beta)| \overset{\text{Hölder}}{\le} \underbrace{\left(\sum \frac{\pi |c_n|^2}{n+1}\right)^{\frac{1}{2}}}_{\|f\|_{A^2}} \cdot \underbrace{\left(\sum \frac{n+1}{\pi} |\beta|^{2n}\right)^{\frac{1}{2}}}_{M_\beta} \Rightarrow |f(\beta)| \le M_\beta \|f\|_{A^2}$$

$\Rightarrow A^2$ is a functional Hilbert space.



Corollary: $f(z) = (f, k_z) = \frac{1}{\pi} \iint_{|z| \le 1} \frac{f(w)}{(1-\bar{z}w)^2} \, dx\,dy$ Ref. Exercise 2.3. (ex 4 will be used later)

Question: Is there any way to "modify" $L^2[-\pi,\pi]$ to make it a functional Hilbert space?

Yes!

core of lecture 2

Exercise 4: $\boxed{\text{Paley–Wiener space}}$ PW

$$PW[0,1] = \left\{ f(z) = \int_0^1 \varphi(t) e^{-2\pi i z \cdot t} \, dt, \ \varphi \in L^2[0,1] \right\}$$

or $[-\pi,\pi]$, e.g. plancherel

不用怕 2π系数 ✓

f is actually an entire function (derivative exists, $\because \varphi \in L^2[0,1]$ compact)

i.e. analytic function over $\mathbb{C}$

With $(f, g) = \int_{-\infty}^{+\infty} f(x) \overline{g(x)} \, dx$

Recall plancherel thm: $\int_{\mathbb{R}} |f|^2 = \int_{\mathbb{R}} |\hat{f}|^2$, $\int_{\mathbb{R}} f \cdot \bar{g} = \int_{\mathbb{R}} \hat{f} \cdot \bar{\hat{g}}$, then

$$(f, g) = \int_0^1 \varphi_f \cdot \overline{\varphi_g}, \quad \text{where} \begin{cases} f(z) = \int_0^1 \varphi_f(t) e^{-2\pi i z \cdot t} \, dt \\ g(z) = \int_0^1 \varphi_g(t) e^{-2\pi i z t} \, dt \end{cases}$$

thanks to the def Hölder is available

$\Rightarrow \because L^2[0,1]$ Hilbert, then $PW[0,1]$ is a Hilbert space, for $\forall z$, $|f(z)| \overset{\text{Hölder}}{\le} \underbrace{\|\varphi\|_{L^2[0,1]}}_{\|f\|_{PW}} \cdot \underbrace{\|e^{2\pi |\operatorname{Im} z| \cdot t}\|_{L^2[0,1]}}_{M_z}$

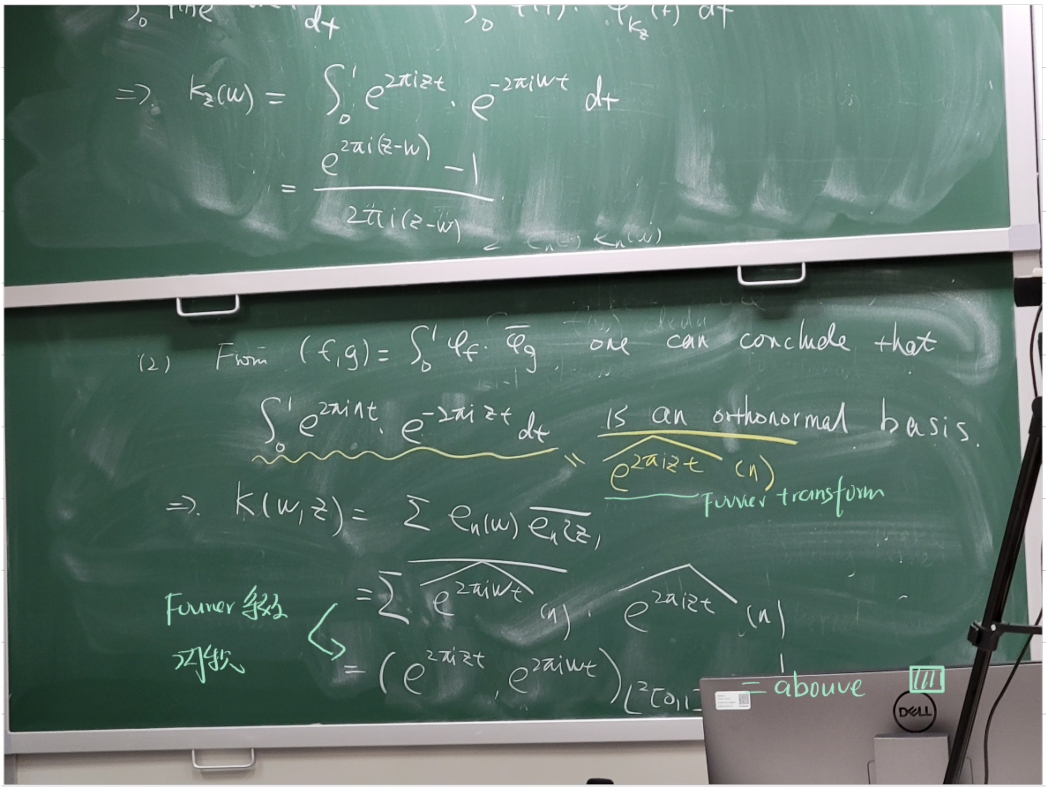
, then $PW[0,1]$ is also a functional Hilbert space.

2 ways to find its reproducing kernel

(1) $f(z) = (f, k_z) = \int_{-\infty}^{+\infty} f(x) \overline{k_z(x)} \, dx$

$$= \int_0^1 \varphi_f e^{-2\pi i z \cdot t} \, dt = \int_0^1 \varphi(t) \cdot \overline{\varphi_{k_z}(t)} \quad \text{(by similar observation by plancherel)}$$

$$\Rightarrow k_z(w) = \int_0^1 e^{2\pi i \bar{z} \cdot t} \cdot e^{-2\pi i w \cdot t} \, dt = \frac{e^{2\pi i (\bar{z}-w)} - 1}{2\pi i (\bar{z}-w)}$$


$\Rightarrow k_z(w) = \int_0^1 e^{2\pi i z t} \cdot e^{-2\pi i w t}\, dt$
$= \frac{e^{2\pi i (z-w)} - 1}{2\pi i (z-w)}$
(2) From $(f,g) = \int_0^1 \varphi_f \cdot \overline{\varphi_g}$ one can conclude that
$\int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i z t}\, dt$ is an orthonormal basis.
$= \widehat{e^{2\pi i z t}}(n)$
Fourier transform
$\Rightarrow K(w,z) = \sum e_n(w) \overline{e_n(z)}$
Fourier 系数
$= \sum \overline{\widehat{e^{2\pi i w t}}(n)} \cdot \widehat{e^{2\pi i z t}}(n)$
$= (e^{2\pi i z t}, e^{2\pi i w t})_{L^2[0,1]}$ = above

Lecture 3: Complete sequences, Coefficient functional, Riesz basis

Lecture 3-2023 年 2 月 23 日天气晴好 ☼，15°C-23°C

主要内容： Riesz basis

其他信息：

$\boxed{X \text{ Banach space}}$, $\mathcal{H}$ Hilbert space

1.5 complete sequences

Def: We say a sequence $\{x_n\}$ is complete in X, if $\forall x \in X$, $\forall \varepsilon > 0$, $\exists c_1, \cdots, c_n$ s.t.

$$\| x - \sum_{i=1}^{n} c_i x_i \| < \varepsilon$$

may depend on ε, hence might not be a basis.

Remark: A complete sequence may not be a basis!

e.g. ℓ^2, $x_1 = e_1$, $x_2 = e_1 + e_2$, $\cdots$, $x_n = e_1 + \cdots + e_n$

An equivalent definition is $\{x_n\}$ is complete iff

"$\mu \in X^*$, $\mu(x_n) = 0, \forall n \Rightarrow \mu = 0$ (Hahn–Banach)

We shall discuss when a complete sequence is a basis

Exercise 1, 7 (Bonus): Show that $\{\frac{1}{x+n}\}_{n=1}^{\infty}$ is complete in $L^2[0,1]$

1.6 the coefficient functionals

If $\{x_1, \cdots\}$ is a basis in X, then $\forall x = \sum_{i=1}^{\infty} c_i x_i$, so $f_n : x \longmapsto c_n$ is a linear functional, and $x = \sum_{n=1}^{\infty} f_n(x) \cdot x_n$

Thm: $f_n \in X^*$, Moreover $1 \leq \|x_n\| \cdot \|f_n\| \leq M$ uniform in n

proof: Since $f_n(x_n) = 1 \Rightarrow \|f_n\| \geq \frac{1}{\|x_n\|}$

$\exists \sum_{n=1}^{\infty} c_n x_n \in X$

Conversely, define $Y = \{ (c_n)_n \}$ with $\| (c_n)_n \|_Y \overset{\text{def}}{=} \sup_n \| \sum_{i=1}^{n} c_i x_i \|_X < \infty$. Y is a Banach space. needs to prove

Define $T : Y \to X$: $(c_n)_n \longmapsto \sum_{n=1}^{\infty} c_n x_n$, linear 1-1 onto, are $\boxed{\text{bounded}}$ as $\|\sum_{n=1}^{\infty} c_n x_n\|_X \leq \sup_n \|\sum_{i=1}^{n} c_i x_i\|$

Now by the open mapping theorem, T is invertible

$\|x_n\| \cdot |f_n(x)| = \|f_n(x) \cdot x_n\| \leq \|\sum_{i=1}^{n} f_i(x) x_i\| + \|\sum_{i=1}^{n-1} f_i(x) x_i\|$ note that $f_n(x) \cdot x_n = \sum_{i=1}^{n} f_i(x) x_i - \sum_{i=1}^{n-1} f_i(x) x_i$

$\leq 2 \cdot \sup_n \|\sum_{i=1}^{n} f_i(x) \cdot x_i\|$

$\leq M \cdot \|\sum_{i=1}^{\infty} f_i(x) \cdot x_i\|$ $\|x\|$

$\Rightarrow \|f_n\| \leq \frac{M}{\|x_n\|}$, with $M = \|T\|^{-1}$, independent in n. ▥

Corollary: Denote $S_n(x) = \sum_{i=1}^{n} c_i x_i$, then $1 \leq \sup_n \|S_n\| < \infty$

proof: the above argument shows

$$\|S_n(x)\| = \|\sum_{i=1}^{n} c_i x_i\| \leq \sup_n \|\sum_{i=1}^{n} c_i x_i\| \leq \|T\|^{-1} \cdot \|x\|.$$ ▥

Theorem: A complete sequence $\{x_n\}$ of non-zero vectors is a basis iff $\exists M > 0$ s.t. $\forall n \leq m$, and scalars $c_1, \cdots, c_m$, we have

$$\|\sum_{i=1}^{n} c_i x_i\| \leq M \cdot \|\sum_{i=1}^{m} c_i x_i\|$$

Proof: $\Rightarrow$ since $n \le m$, then $(\sum_{i=1}^n c_i x_i)$ 为 $\sum_{i=1}^m c_i x_i$ 的部分和, then by the abouve Corollary

$\sup_n \|\sum_{i=1}^n c_i x_i\| < \infty$, as $S_n(\sum_{i=1}^m c_i x_i) = \sum_{i=1}^n c_i x_i$

$\Leftarrow$ Since $\{x_n\}$ is complete, we have $\sum_{i=1}^n c_i x_i \to x$, as $n \to \infty$.

From $\|(c_{kn} - c_{km}) x_k\| \le \|\sum_{i=1}^k (c_{in} - c_{im}) x_i\| + \|\sum_{i=1}^{k-1} (c_{in} - c_{im}) x_i\|$

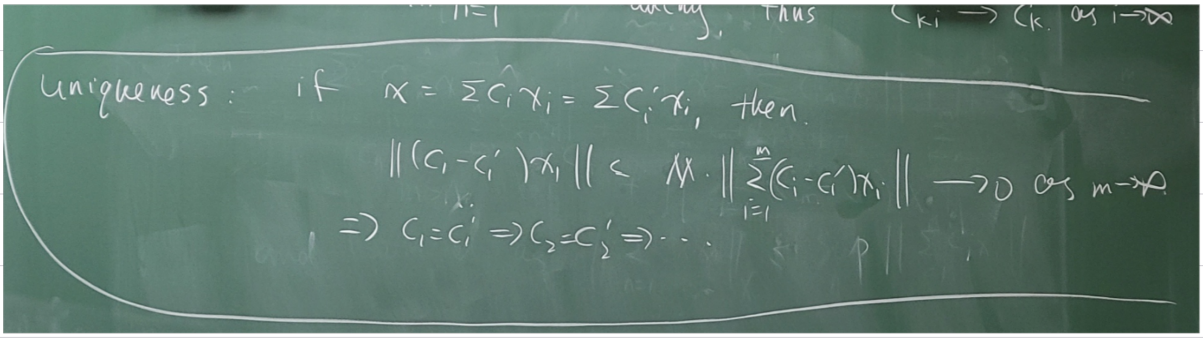
Say $m > n > k$, with k fixed. $\le M \|\sum_{i=1}^m (\tilde{c}_{in} - c_{im}) x_i\|$, where $\tilde{c}_{in} = \begin{cases} c_{in}, & i \le n \\ 0, & i > n \end{cases}$

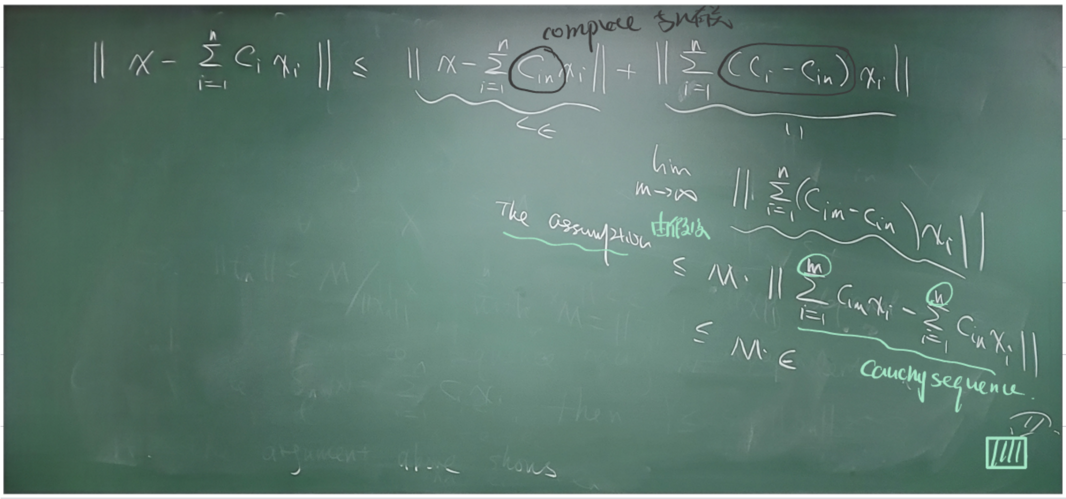
$$\|\sum_{i=1}^m (\tilde{c}_{in} - c_{im}) x_i\| = \|\sum_{i=1}^n c_{ik} x_i - \sum_{i=1}^m c_{im} x_i\|$$

$\Rightarrow \forall k$ $\{c_{ki}\}_{i=1}^\infty$ is Cauchy thus $c_{ki} \to c_k$, as $i \to \infty$

但是可数, 不存在问题!

先证 uniqueness





接上

Exercise 1.3. (HW)

Section 1.7: Duality ($\{x_n\} \subset X$, $\{f_n\} \subset X^*$, $f_n(x_m) = \delta_{nm}$)

basis; coefficient functional.

Observation: $\{x_n\}$ is a basis $\not\Rightarrow$ $\{f_n\}$ is a basis of X^*

e.g. Exercise 1: $(\ell^1)^* = \ell^\infty$, non-separable ?

still wrong even if X^* is assumed separable

Exercise 1: $C[0,1]$ with Schauder basis.

Theorem: If $\{x_n\}$ is a basis for X, then $\{f_n\}$ is a basis for $[f_n] \overset{\text{def}}{=} \text{span}\{f_n\}$

proof: Consider $S_n^*(f)(x) = f(S_n(x)) = f(\sum_{i=1}^n f_i(x) x_i) = \sum_{i=1}^n f_i(x) \cdot f(x_i)$

$\Rightarrow S_n^*(f) = \sum_{i=1}^n f(x_i) f_i \xrightarrow{\text{to show}} f \in [f_n]$

$\forall f \in [f_n]$, $\forall \varepsilon > 0$, $\exists g = \sum_{\text{finite}} c_i f_i$ s.t. $\|f - g\| < \varepsilon$

$\|S_n^* f - f\| \le \underbrace{\|S_n^*(f-g)\|}_{\le M\|f-g\| < M\cdot\varepsilon} + \underbrace{\|S_n^* g - g\|}_{=0,\ \text{when } n \text{ is large}} + \|f-g\|_{<\varepsilon}$

uniqueness: if $0 = \sum c_i f_i$, then $0 = \sum c_i f_i(x_n) = c_n$, $\forall n$ $\boxed{\text{///}}$

Theorem 6: If X is reflexive, then $\{f_n\}$ is a basis for X^*.

Proof: It suffices to prove $\{f_n\}$ is complete in X^*

$\forall x \in (X^*)^* = X$, if $\underbrace{x(f_n)}_{= f_n(x)} = 0$, $\forall n \Rightarrow x = 0$ $\boxed{\text{///}}$

Now consider Hilbert space $\mathcal{H}$ (↗总可以找正交补), we say $\{x_n\}$, $\{y_n\}$ are bi-orthogonal if $(x_n, y_m) = \delta_{n,m}$

Remark: ① There exists a biorthogonal sequence of $\{x_n\}$, iff $\{x_n\}$ is minimal, i.e. $\forall n$ $x_n \notin \overline{\operatorname{span}}_{m\neq n}\{x_m\}$

② If $\{x_n\}$ is minimal, then its biorthogonal sequence is unique iff $\{x_n\}$ is complete.

③ If $\underbrace{\{x_n\} \text{ is a basis}}_{\text{minimal + complete}}$, so is its $\underbrace{\text{biorthogonal basis}}_{\text{unique}}$ $\{y_n\}$

④ Let $\{f_n\}$, $\{g_n\}$ be bi-orthogonal basis, then $x = \sum (x, f_n) g_n = \sum (x, g_n) f_n$

$(x = \sum c_n f_n$, then $(x, g_m) = c_m)$

HW: Exercise 1.4

Main tool: Riesz basis (may not be orthogonal, but not too away from orthogonal)

↓ frequently used later in this course.

Section 1.8: Riesz Bases

Def: 2 bases are equivalent for a Banach space X, if $\exists$ a bounded invertible linear operator

$T: X \to X$ s.t. $T x_n = y_n$, $\forall n$

Thm: An equivalent def is

$\sum_{n=1}^\infty c_n x_n$ is convergent $\Leftrightarrow \sum_{n=1}^\infty c_n y_n$

Proof: "$\Rightarrow$" by definition and T

"$\Leftarrow$" let $T(\sum_{n=1}^\infty c_n x_n) = \sum_{n=1}^\infty c_n y_n$, well-defined, 1-1, onto

Consider $T_n(\sum_{i=1}^\infty c_i x_i) = \sum_{i=1}^n c_i y_i = S_n(\sum_{i=1}^\infty c_i y_i)$

$\Rightarrow \forall x$ $\sup_n |T_n(x)| \le \sup_n \|S_n\| \cdot \|y\| < \infty$

and $T_n(x) \to T(x)$

Now by the Banach–Steinhaus (共鸣定理) $\Rightarrow \|T\|<\infty$, ▯

Thm 8: In $\mathcal{H}$, equivalent bases $\{x_n\}, \{y_n\}$ have equivalent bi-orthogonal sequences $\{f_n\}, \{g_n\}$

Proof: $Tx_n = y_n$, claim $T^* g_n = f_n$

to see this $(T^* g_n, x_m) = (g_n, Tx_m) = (g_n, y_m) = \delta_{n,m}$. ▯

Def: A basis for H is called a $\boxed{\text{Riesz basis}}$, if it is equivalent to an orthonormal basis

$(\underset{\uparrow \text{invertible}}{T} e_n = f_n)$

Remark: ① $\frac{1}{\|T^{-1}\|} \le \|f_n\| \le \|T\|$ (so $\{n \cdot e_n\}$ is not a Riesz basis)

② $\{f_n\}$ is a Riesz basis $\Rightarrow \left\{\frac{f_n}{\|f_n\|}\right\}$ is a Riesz basis

(pf: $f_n \Leftrightarrow e_n \Leftrightarrow \|f_n\| e_n$)

③ If $\{f_n\}$ is a Riesz basis, so is its biorthogonal sequence (Thm 8)

④ $\{|t|^{\alpha} e^{int}\}_{n=-\infty}^{+\infty}$, $0<\alpha<\frac{1}{2}$, is a bounded basis but not a Riesz basis (Babenko, 1948)

Thm: In $\mathcal{H}$, TFAE (1)→(5)

(1) $\{f_n\}$ is a Riesz basis

(2) $\exists$ an equivalent inner product $(\cdot,\cdot)$ (i.e. $\|f\|_1 \approx \|f\|$)

under which $\{f_n\}$ is an orthonormal basis.

commonly used

$\boxed{(3)}$ $\{f_n\}$ is complete and $\exists A, B>0$, s.t. $\forall n>0$, $c_1, \cdots, c_n$

$$A \cdot \sum_{i=1}^{n} |c_i|^2 \le \left\|\sum_{i=1}^{n} c_i f_i\right\|^2 \le B \cdot \sum_{i=1}^{n} |c_i|$$

(4) $\{f_n\}$ is complete and the Gram matrix operator $(f_i, f_j)_{i,j}$ generates a

bounded invertible linear operator in ℓ^2, G in ℓ^2, where

$$G\underbrace{(c_n)_n}_{\substack{\text{sequence} \\ n\text{为指标}}} = \left(\sum_n \overline{(f_m, f_n)}\, c_n\right)_{\underline{m}\ m\text{为指标}}$$

(5) $\{f_n\}$ is complete, and posseses a complete bi-orthogonal sequence s.t. $\forall f \in \mathcal{H}$

$\sum |(f, f_n)|^2 < \infty$, $\sum |(f, g_n)|^2 < \infty$

Lecture 4: cont. Equivalent condition of Riesz basis, Paley-wiener criterion

Lecture 4-2023 年 2 月 28 日天气晴好 ☼，14°C-23°C

主要内容： application of riesz basis

其他信息：

02-28

cont. Equivalence of Riesz basis. 反例: $(C[0,1])^*$ is not separable e.g. $\|\delta_x-\delta_y\|=1,\ \forall x\neq y$

↓

见 duality sec1.7 Exercise 1 (quite difficult!)

Thm: In $\mathcal{H}$, TFAE (1)→(5)

(1) $\{f_n\}$ is a Riesz basis

(2) $\exists$ an equivalent inner product $(\cdot,\cdot)_1$ (i.e. $\|f\|_1\approx\|f\|$) under which $\{f_n\}$ is an orthonormal basis.

commonly used

$\boxed{(3)}$ $\{f_n\}$ is complete and $\exists A,B>0$, s.t. $\forall n>0, c_1,\cdots,c_n$

$$A\cdot\sum_{i=1}^n |c_i|^2 \le \Big\|\sum_{i=1}^n c_i f_i\Big\|^2 \le B\cdot\sum_{i=1}^n|c_i|^2$$

(4) $\{f_n\}$ is complete and the Gram matrix operator $(f_i,f_j)_{i,j}$ generates a bounded invertible linear operator in ℓ^2, G in ℓ^2, where

$$G: \underbrace{(c_n)_n}_{\text{sequence, n为指标}} = \Big(\sum_n (f_m,f_n)\, c_n\Big)_{\underline{m}\text{为指标}}$$

(5) $\{f_n\}$ is complete, and posseses a complete bi-orthogonal sequence $\{g_n\}$ s.t. $\forall f\in\mathcal{H}$

$\sum|(f,f_n)|^2<\infty$, $\sum|(f,g_n)|^2<\infty$

difficult part

$$\begin{tikzcd} \boxed{(5)} \arrow[r, shift left] & (1) \arrow[l, shift left] \arrow[r] \arrow[dl] & (4) \arrow[dl] \\ (2) \arrow[r] & (3) \arrow[u] & \end{tikzcd}$$

proof: First the road map of the proof:

(1)→(2): $Tf_n=e_n$, then define an inner product $(f,g)_1 \overset{def}{=} (Tf,Tg)$ (omit: checking the inner product is equivalent)

(2)→(3): From (2): $\|\sum_{i=1}^n c_i f_i\|_1 \approx \|\sum_{i=1}^n c_i f_i\|$ (new norm from (2))

$\|\sum_{i=1}^n c_i f_i\|_1 = \sqrt{\sum_{i=1}^n |c_i|^2}$ under $(\cdot,\cdot)_1$, $\{f_n\}$ is an orthonormal basis.

(3)→(1): let $Te_n=f_n \Rightarrow \{f_n\}$ is a riesz basis

So far we have shown that $(1)\to(2)\to(3)\to(1)$, Now for (1)→(4)

(1)→(4) completeness ✓, $Te_n=f_n$, then $(f_i,f_j)=(e_i,T^*Te_j)$, so

$$\Big\|\Big(\sum_j \overline{(f_i,f_j)}\, c_j\Big)_i\Big\|_{\ell^2} = \Big\|\Big(f_i,\sum_j c_j f_j\Big)_i\Big\|_{\ell^2}$$

$$= \Big\|\Big\langle e_i, T^*T\sum_j c_j e_j\Big\rangle_i\Big\|_{\ell^2}$$

$$= \Big\|T^*T\sum_j c_j e_j\Big\|_{\mathcal{H}} \overset{由T的性质}{\approx} \Big\|\sum_j c_j e_j\Big\| = \sqrt{\sum_j |c_j|^2}$$

Hence we construct a bounded invertible linear operator in ℓ^2, namely G.

(4)→(3): $\|\sum_{i=1}^n c_i f_i\|^2 = (\sum_{j=1}^n c_j f_j, \sum_{i=1}^n c_i f_i) = \Big\langle (c_i)_i, \underbrace{\sum_j \overline{(f_i,f_j)}\, c_j}_{\text{Gram matrix operator } G}\Big\rangle$

Recall that in Functional Analysis: If operator $G>0$ (i.e. $\langle Gf,f\rangle>0$), then $\exists P^2=G$, s.t. $(Gf,f)=\|Pf\|^2$

Now by the above, $\Rightarrow G\ge 0 \Rightarrow \exists P^2=G$, then $\|\sum c_i f_i\|^2 = \|P(c_i)\|^2 \approx \|(c_i)\|_{\ell^2}^2$

Now we show that (1), (5) are equivalent.

(1) → (5): $Te_n = f_n$, $\{f_n\}$ is a Riesz basis ⇒ so is $\{g_n\}$ $T^* g_n = e_n$

$$\Rightarrow \forall f, \quad f = \sum \langle f, f_n\rangle g_n = \sum \langle f, g_n\rangle f_n$$

$$\| \quad (T^*)^{-1} \sum \langle f, f_n\rangle e_n, \qquad = T\sum \langle f, g_n\rangle e_n$$

since T is $\overset{\text{bounded}}{\text{invertible}}$ $\Rightarrow \|f\|^2 \approx \sum |\langle f, f_n\rangle|^2 \approx \sum |\langle f, g_n\rangle|^2$.

a little harder than (1) → (5)

(5) → (1)

$$\text{"}\sum |\langle f, f_n\rangle|^2 < \infty\text{"} \Longrightarrow \exists C > 0 \text{ s.t. } \sum |\langle f, f_n\rangle|^2 \le C^2 \|f\|^2$$

$\Sigma^1, \Sigma^2, \ldots$，取部分和构建算子族，利用 uniform boundedness thm

Similarly $\sum |\langle f, g_n\rangle|^2 \le D^2 \|f\|^2$

Define $\begin{cases} Sf_n = e_n, & \text{densely defined in } \{\sum_{\text{finite}} c_i f_i\} \\ Tg_n = e_n & \{\sum_{\text{finite}} c_i g_i\} \end{cases}$

⇒ S, T can be extended to bounded operators with $\|S\| \le C$. ($\|T\| \le D$)

Recall that $Sf_n = e_n \Rightarrow S^* e_n = g_n$

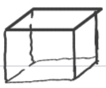
$Tg_n = e_n \Rightarrow T^* e_n = f_n$

$T^* S = ST^* = \mathrm{Id}$. ▨

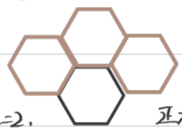
Exercise 3.5 (use Equivalent conditions 1~5)

Exponential basis for $\underset{\text{domain}}{\Omega} \subseteq \mathbb{R}^n$ (Application of Riesz basis)

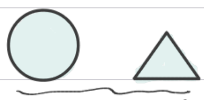
Example



Ω, $\{e^{2\pi i n\cdot x}\}_{n\in\mathbb{Z}^n}$ is an orthonormal basis



$n=2$. 正六边形. Fuglede conjecture (1974)



No basis!

"Ω has orthonormal basis iff Ω tiles $\mathbb{R}$ by translation"

FALSE! , $n \ge 3$ (Start by Terry Tao), If Ω is convex, then the conjecture is True (2019)

Question 2: Is there exponential $\boxed{\text{Riesz}}$ basis on Ω
$\{e^{2\pi i \lambda_n \cdot x}\}_n$

The first example of non-existence was given last year, we may leave it to the summer,

Next we shall discuss Paley-Wiener's motivation

Question of Paley-Wiener: If perturbate $n \in \mathbb{Z}$ to λ_n, is $\{e^{2\pi i \lambda_n \cdot x}\}$ still a basis of $L^2[0,1]$

Roughly speaking, if $\{x_n\}$ is a basis, and $\{y_n\}$ is close to $\{x_n\}$, then $\{y_n\}$ is a basis equivalent to $\{x_n\}$.

Key fact: $\|I-T\|<1 \Rightarrow T$ is bounded invertible

Section 1.8: The stability of basis in Banach (with basis $\{x_n\}$)

Thm 10: If $\exists\, 0<\lambda<1$, s.t. $\forall n$, $\forall c_1,\cdots,c_n$,

$$\|\sum_{i=1}^n c_i(x_i-y_i)\| \le \lambda \|\sum_{i=1}^n c_i x_i\|,$$

then $\{y_n\}$ is a basis, equivalent to $\{x_n\}$.

Remark:

① $\lambda<1$ is necessary. i.e. $y_n=0$, $y_n=\frac{x_n}{n}$ (see Exercise 1, for a stronger version)

Proof: let $T(\sum c_i x_i)=\sum c_i(x_i-y_i)$, well-defined, bounded in X. $\|T\|\le\lambda<1$

then $\|I-(I-T)\|<1$

$\Rightarrow I-T$ is invertible, and $(I-T)x_n=y_n$. ▥

Corollary: Let $\{f_n\}$ be coefficient functionals for $\{x_n\}$, if $\sum\|f_n\|\cdot\|x_n-y_n\|<1$, then $\{y_n\}$ is a basis equivalent to $\{x_n\}$

proof: $\|\sum_{i=1}^n c_i(x_i-y_i)\| = \|\sum_{i=1}^n f_i(x)(x_i-y_i)\| \le \sum_{i=1}^n \|f_i(x)\|\cdot\|x_i-y_i\|$

$$\le \underbrace{\left(\sum_{i=1}^n \|f_i\|\cdot\|x_i-y_i\|\right)}_{<1,\text{ by condition}} \cdot \underbrace{\|x\|}_{\|\sum_{i=1}^n c_i x_i\|}$$

then by thm 10, $\{y_n\}$ is a basis equivalent to $\{x_n\}$. ▥

Cor of Cor above (Thm Krein-Milman-Rutman)

$\exists\, \varepsilon_n>0$, s.t. $\{y_n\}$ is a basis equivalent to $\{x_n\}$, whenever $\|y_n-x_n\|<\varepsilon_n$

Application: Recall in Lecture 1, we've constructed $\{e_n\}$ "$\wedge$" basis of $C[0,1]$.

If a Banach Space has a basis, then every dense subset contains a basis.

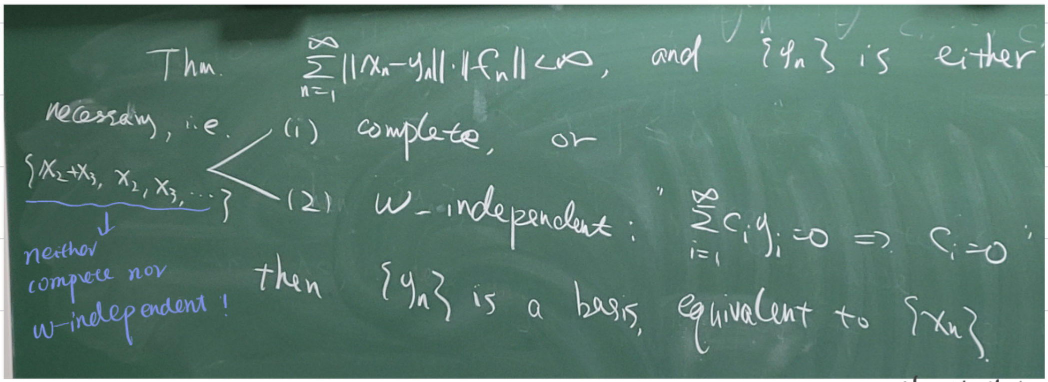
In particular, $C[0,1]$ has a polynomial basis.

Thm: $\sum_{n=1}^\infty \|x_n-y_n\|\cdot\|f_n\|<\infty$, and $\{y_n\}$ is either

(1) complete, or

(2) ω-independent: $\sum_{i=1}^\infty c_i y_i=0 \Rightarrow c_i=0$

then $\{y_n\}$ is a basis equivalent to $\{x_n\}$



Proof: By previous thm (or previous corollary), one can conclude that $\exists N>0$ (or $0+\cdots+0+\sum \|f_n\|\|x_n-y_n\| < 1$) s.t.

$\{x_1, \cdots, x_{N-1}, y_N, y_{N+1}, \cdots\}$ is a basis equivalent to $\{x_n\}$, then consider

$\bar{X} = X/\text{span}\{y_N, y_{N+1}, \cdots\}$, space of basis $\bar{x}_1, \cdots, \bar{x}_{N-1}$ (or finite dimensional space.)

$(N-1)$-dim Banach space.

$$\|x\|_{\bar{X}} \overset{\text{def}}{=} \inf_{y-x \in \text{span}\{y_N, y_{N+1}, \cdots\}} \|y\|$$

assumption (1) $\Rightarrow \bar{y}_1, \cdots \bar{y}_{N-1}$ complete in $\bar{X}$

$\Rightarrow \bar{y}_1, \cdots \bar{y}_{N-1}$ is a basis for $\bar{X}$

then assumption (2) $\Rightarrow \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{N-1}$ is linearly independent

$\Rightarrow \bar{y}_1, \cdots, \bar{y}_{N-1}$ is a basis for $\bar{X}$.

Therefore $\forall y \in X$. $\exists!\, c_1, \cdots, c_{N-1}$ s.t. $y - \sum_{i=1}^{N-1} c_i y_i \in \text{span}\{y_N, \cdots\}$

$$y - \sum_{i=1}^{N-1} c_i y_i = \sum_{i=N}^{\infty} c_i y_i$$

$\Rightarrow y = \sum_{i=1}^{N} c_i y_i$, unique. ////

Exercise 1.

Above are results in Banach space. all results in Banach space remain valid in Hilbert space.

but Hilbert space has extra structure e.g. $\|\sum c_i e_i\|^2 = \sum |c_i|^2$

Thm 13 $\|\sum c_i (e_i - f_i)\| \le \lambda \sqrt{\sum |c_i|^2}$ $(0<\lambda<1)$ $\Rightarrow \{f_i\}$ is a Riesz basis.

Paley-Wiener criterion

Thm 14 (Kadec's $\frac{1}{4}$-theorem) If $\lambda_n \in \mathbb{R}$, and $|\lambda_n - n| \le L < \frac{1}{4}$, then $\{e^{i\lambda_n t}\}_n$ is a Riesz basis

for $L^2[-\pi, \pi]$, moreover $\frac{1}{4}$ is sharp. with example $\lambda_n = \begin{cases} n - \frac{1}{4}, & n>0 \\ 0, & n=0 \\ n+\frac{1}{4}, & n<0 \end{cases}$

Sketch of proof: $\forall \sum_j |c_j|^2 < 1$. denote $S_n = \lambda_n - n$, then

$\|\sum c_i e^{int}(1 - e^{iS_n t})\| \le 1 - \cos(\pi L) + \sin(\pi L) < 1$

and the expansion of $1-e^{iSnt}$ relies on the orthonormal basis 正弦系

$\{1, \cos(nt), \sin(n-\frac{1}{2})t, \cdots\}$ for $L^2[-\pi,\pi]$ exercises in book

Remark: by considering a different orthonormal basis for $L^2[-\pi,\pi]$, Duffin and Eachus proved that $\{e^{i\lambda_n t}\}$ is a Riesz basis if $\lambda_n \in \mathbb{C}$, $|\lambda_n - n| \le L < \frac{\log 2}{\pi}$

Another merit of Hilbert space (Cauchy-Schwarz)

Notice that $\|\sum c_i(e_i - f_i)\| \le \sum \|c_i\| \cdot \|e_i - f_i\| \overset{\text{Cauchy}}{\le} \underbrace{(\sum \|c_i\|^2)^{\frac{1}{2}}}_{\|\sum_i c_i e_i\|} \cdot (\sum \|e_i - f_i\|^2)^{\frac{1}{2}}$

So $\sum_i \|e_i - f_i\|^2 < 1 \Rightarrow \{f_i\}$ is a Riesz basis.

Similar to thm 13, $\sum \|e_i - f_i\| < \infty$ + $\{f_i\}$ is either complete or ω-independent $\Rightarrow \{f_i\}$ is a Riesz basis (Bari basis)

Application: $\{\sqrt{2}\cos(n\pi t) + \frac{S_n(t)}{n+1}\}$

With $|S_n(t)|$ bounded is a Riesz basis for $L^2[0,1]$.

The Paley-Wiener criterion, namely $\|\sum c_i(e_i - f_i)\| \le \lambda\sqrt{\sum |c_i|^2}$

states that the operator $T: e_n \to f_n$ is an isomorphism, $\|I - T\| < 1$. In fact, every Riesz basis can be obtained in this way.

Thm $\{f_n\}$ is a Riesz basis for $\mathcal{H}$, then $\exists$ an orthonormal basis $\{e_n\}$ 去找$\{e_n\}$, an isomorphism T, and $p > 0$, s.t.

$Te_n = \underbrace{p \cdot f_n}_{= g_n}$, and $\|I - T\| < 1$

Heavily relies on Functional analysis

Proof: Since $\{f_n\}$ is a Riesz, $\exists$ an orthonormal basis $\{\phi_n\}$, an isomorphism $S: \phi_n \mapsto f_n$ and $A, B > 0$

s.t. $A\sum |c_n|^2 \le \|S(\sum c_n \phi_n)\|^2 \le B \sum |c_n|^2$

Let $p = \frac{2}{\sqrt{A} + \sqrt{B}}$, $g_n = p f_n$, then

$(1-\lambda)\sqrt{\sum |c_n|^2} \le \|\sum_n c_n g_n\| \le (1+\lambda)\sqrt{\sum |c_n|^2}$, and $\lambda = \frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}} < 1$

then it suffices to show that $\exists$ an orthonormal basis $\{e_n\}$ s.t.

$\|\sum c_i(e_i - g_i)\| \le \lambda\sqrt{\sum_i |c_i|^2}$

$S = \underset{\text{unitary}}{U} \underset{\text{positive}}{P}$ (polar decomposition in Functional analysis)

and $e_n = U\phi_n$, since P is self adjoint, so $I - P$ is also self-adjoint, and

$$\|I-P\| = \sup_{\|f\|=1} |((I-P)f,f)| = \sup_{\|f\|=1} \Big| \|f\| - \underbrace{(Pf,f)}_{>0} \Big|$$

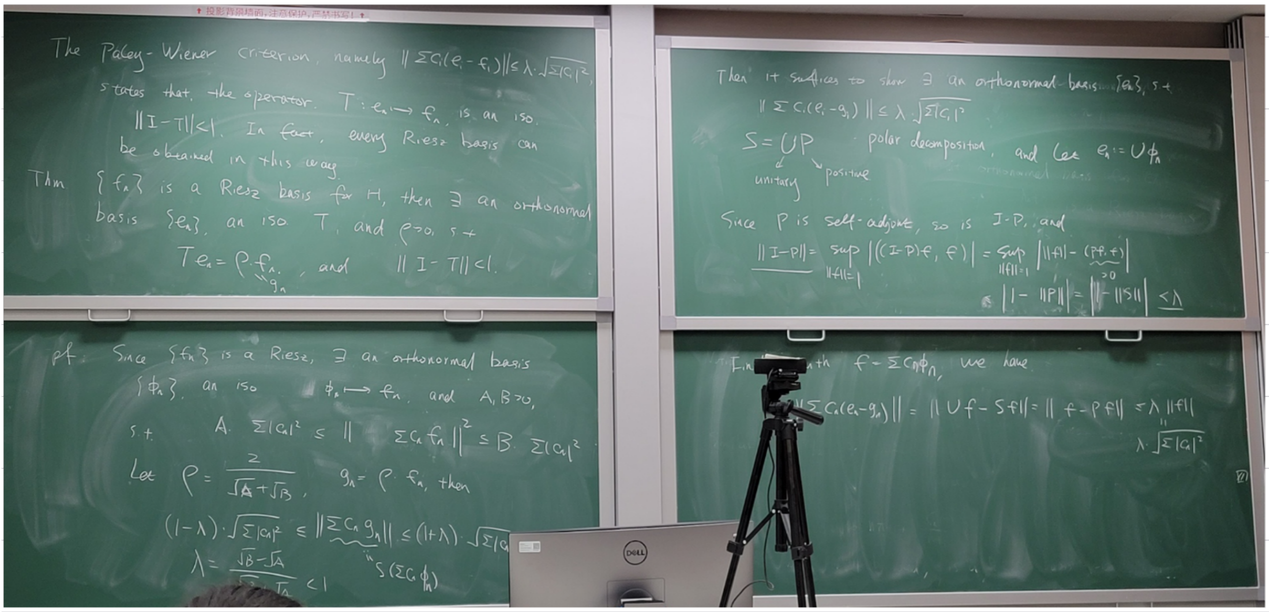
$$\leq \big|1-\|P\|\big| = \big|1-\|S\|\big| < \lambda.$$

↑ 可对f取

Finally, with $f = \sum c_n \phi_n$, we have $\|\sum c_n(e_n - g_n)\| = \underline{\|Uf - Sf\|}$ $\quad$ S: 行为 $\phi_n \to g_n$!

$$= \|f - Pf\|$$

$$\leq \lambda\|f\| = \lambda\sqrt{\sum|c_i|^2} \qquad \square$$



End of Chapter 1.

Thursday: about Assignment.

Next chapter mainly use Complex analysis.

Lecture 5: Problem Set Discussion-1, Chapter 2: Entire functions of Exponential Type

Lecture 5-2023 年 3 月 2 日天气晴好 ☼，15°C-23°C

主要内容：几个 Factorization theorem，其实都是为了刻画增长性

其他信息：

03-02

1.2 Q4 $f \in C[a,b]$. Show that $\exists$ polynomials $P_1, P_2, \dots$ s.t. $f = \sum P_n$, and the series is convergent (depend on f) absolutely and uniformly.

Pf. $\exists$ Q_n s.t. $\|f - Q_n\|_{L^\infty} < \frac{1}{2^n}$

Let $P_0 = Q_0$, $P_n = Q_n - Q_{n-1}$

then $f - \sum_{n=0}^{N} P_n = f - Q_N$

⬆ 投影背景墙面，注意保护，严禁书写！⬆

1.3 (12) An orthonormal sequence $\{e_n\}$ in $L^2[a,b]$ is complete iff $\sum_{n=1}^{\infty} \left| \int_a^x e_n(t)\,dt \right|^2 = x - a$, $\forall\, x \in [a,b]$

"$\Rightarrow$" $\sum |(1_{[a,x]}, e_n)|^2 = \|1_{[a,x]}\|^2_{L^2[a,b]}$

"$\Leftarrow$" $1_{[a,x]} = \sum_{n=1}^{\infty} (1_{[a,x]}, e_n)\, e_n$, $\forall\, x$

then $1_{[x,y]} = \sum_{n=1}^{\infty} (1_{[x,y]}, e_n)\, e_n$

then $f = \sum_{n=1}^{\infty} (f, e_n)\, e_n$, $\forall$ simple function

which means $\text{span}(e_n)$ is dense in L^2, so $\overline{\text{span}(e_n)} = L^2$ 闭子空间

3. A function K defined on $S \times S$ is called a **positive matrix** if for each positive integer n and each choice of points $t_1, \dots, t_n$ from S the quadratic form

$$\sum_{j=1}^{n} \sum_{i=1}^{n} K(t_i, t_j)\xi_i \bar{\xi}_j$$

is positive definite.

(a) Show that the reproducing kernel of a functional Hilbert space is a positive matrix.

(b) Show that if K is a positive matrix, then there is a functional Hilbert space whose reproducing kernel is K.

1.4 ③ K on $S\times S$ is called a positive matrix, if $\forall\, n$, $\forall\, t_1,\cdots,t_n\in S$, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{n}K(t_i,t_j)\xi_i\bar{\xi}_j \text{ is positive definite}$$

(a) Show that the reproducing kernel is a positive matrix

$$K(x,y)=\sum_{n=1}^{\infty}e_n(x)\overline{e_n(y)}$$

$$\Rightarrow \underbrace{\sum_{i,j=1}^{n}K(t_i,t_j)\xi_i\bar{\xi}_j}_{\text{利用 reproducing kernel 的定义}} = \sum_{i,j,m}e_m(t_i)\overline{e_m(t_j)}\xi_i\bar{\xi}_j = \sum_m\left|\sum_{i=1}^{n}e_m(t_i)\xi_i\right|^2$$

修

$$\sum K(t_i,t_j)\xi_i\bar{\xi}_j = \left\|\sum k_{t_i}\xi_i\right\|^2$$

$$= \sum (k_{t_j},k_{t_i})\xi_i\bar{\xi}_j$$

$$= \left\|\sum \xi_i k_{t_i}\right\|^2$$

(b) Show that if K on $S\times S$ is a positive matrix, then $\exists$ a functional Hilbert space ... reproducing kernel is K

Let $H=\operatorname{span}\{K_y=K(\cdot,y),\ y\in S\}$, with

$$(K_{y_1},K_{y_2})=K(y_2,y_1)$$

Stein's book, Complex Analysis, Chapter 2, Thm 5.2 — complex analysis

$\{f_n\}$ holo, $f_n\rightrightarrows f$ in every compact subset of Ω, then f is holo. in Ω.

pf. By thm 5.1 (利用thm5.1 triangle 证明) $\int_{\text{triangle}} f=0 \Rightarrow f$ is holo for every triangle.

Weierstrass thm. used later in the next chapter. Stein的第五章

Show that $\left\{\frac{1}{x+n}\right\}_{n=1}^{\infty}$ is complete in $L^2[0,1]$.

Pf. It suffices $t^m\in\operatorname{span}\left\{\frac{1}{x+n}\right\}$, $\forall\, m=0,1,2,\cdots$

First $\frac{n}{x+n}\to 1$

Then, by $\boxed{\text{induction}}$ (利用induction), one can see $\frac{x^m}{x+n}\in\operatorname{span}$, x^m

$m=0$ ✓

$$\frac{x^{m+1}}{x+n}=\frac{x^m(x+n)-nx^m}{x+n}=x^m-\frac{n\cdot x^m}{x+n}\in\operatorname{span} \quad \text{by induction hypothesis}$$

$$x^{m+1}=\lim_{n\to\infty}\frac{n\cdot x^{m+1}}{x+n}$$

□

End of the QA of exercises

Chapter 2: Entire Functions of Exponential Type.

we might focus on some specific results in Complex Analysis

Why Entire? say in $C[a,b]$. if $\{e^{i\lambda_n t}\}$ is not complete, then $\exists\, \mu \in C^*[a,b]\setminus\{0\}$, s.t.

closely related to fourier transform

$$\hat{\mu}(\lambda_n) \overset{\text{def}}{=} \underbrace{\int_a^b e^{-i\lambda_n t}\, d\mu(t)}_{\text{Fourier transform}} = 0$$

$\hat{\mu}(z)$ is entire

part 1: The classical Factorization Theorems

stein's book is released later than this one, maybe better than this book

Jensen's formula: f is holomorphic in B_R, continuous in the boundary, $f \neq 0$ in $\{0\} \cup \partial B_R$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{k=1}^{n} \log\left(\frac{R}{|z_k|}\right)$$

where $z_1, \cdots, z_n$ are zeros with multiplicity.

Def: An entire function is of exponential type B, if $|f(z)| \le A\cdot e^{B|z|}$, for some $A, B > 0$.

We say that it has finite order, if $|f(z)| \le A\cdot e^{B|z|^\rho}$, for some $A, B, \rho > 0$

the "smallest" ρ is called the order of f: denote by $\operatorname{ord}(f)$

Note that exponential type $\neq$ order 1, e.g. $e^{|z|\cdot\log|z|}$.

Thm: Denote $n(r) \overset{\text{def}}{=}$ # of zeros in B_r, then $n(r) = O(r^{\operatorname{ord}(f)+\varepsilon})$, $\forall \varepsilon > 0$

proof: By Jensen's formula.

Def: Canonical factor of order k:

$$E_0(z) = 1 - z, \quad E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}$$

Weierstrass factorization thm:

f: entire, not identically 0, then $f(z) = z^m \cdot e^{g(z)} \cdot \prod_{n=1}^{\infty} E_n(z/z_n)$, 除法

where g is entire, $z_1, \cdots$, are non-zeros with multiplicity.

Hadamard Factorization Thm:

If f has finite order, denote $k \overset{\text{def}}{=} [\operatorname{ord}(f)]$, then

$f(z) = e^{p(z)} \cdot z^m \cdot \prod_{n=1}^{\infty} E_k(z/z_n)$, where $p(z)$ is a polynomial of order $\le k$

Example: $\sin(\pi z) = \pi z \cdot \prod_{n\neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$, mainly usage of def of order.

Part 2: Restriction Along a Line

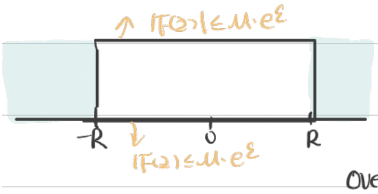
quite useful, even in research!

Recall the Hadamard 3-Lines lemma

f holomorphic and bounded in $\{0 < \operatorname{Im} z < 1\}$, continuous in the boundary and $|f(x)|, |f(x+i)| \leq M$, $\forall x$, then $|f(z)| \leq M$ in this strip.

边界上的估计

Proof: let $F(z) = e^{-\varepsilon z^2} f(z)$, analytic, $|F(z)| = e^{-\varepsilon x^2 + \varepsilon y^2} \cdot |f(z)| \to 0$, as $|x| \to \infty$

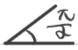


$\exists R > 0$, s.t. $|F| \leq M$, outside $\{|x| < R\}$, in $[-R, R] \times [0, 1]$, we can apply maximum principle to conclude that $|F(z)| \leq M \cdot e^{\varepsilon}$

Overall, $|f(z)| \leq e^{\varepsilon(x^2 - y^2) + \varepsilon} \cdot M$, $\to M$ when $\varepsilon \to 0$. ▥

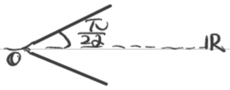
the above result will be used frequently later.

Thm 6 (Phragmén–Lindelöf)



f analytic, continuous in the boundary, $|f(z)| \leq M$ in the boundary, and has order $< \alpha$ inside the sector, then $|f(z)| \leq M$, in this sector.

Proof: First assume



, and let $g(z) = e^{-\varepsilon z^{\gamma}} f(z)$, where $\operatorname{order}(f) < \gamma < \alpha$

gamma

$1^{\gamma} = 1$, then with $z = re^{i\theta}$.

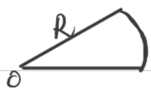
Interior: $|g(z)| = e^{-\varepsilon r^{\gamma} \cos(\gamma\theta)} |f(z)|$, since $\gamma < \alpha$, $\gamma\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\cos(\gamma\theta) \geq \cos(\gamma \cdot \frac{\pi}{2\alpha}) > 0$

$\leq e^{-\varepsilon r^{\gamma} c} \cdot \underbrace{A \cdot e^{B \cdot r^{\operatorname{ord}(f) + \varepsilon}}}_{\text{exponential of } f} \to 0$, when $r \to \infty$. if ε small enough s.t. $\gamma > \operatorname{ord}(f) + \varepsilon$

On boundary: i.e. $\theta = \pm \frac{\pi}{2\alpha}$, $|g(z)| = e^{-\varepsilon r^{\gamma} \cos(\frac{\gamma\pi}{2\alpha})} |f(z)| \leq e^{-\varepsilon r^{\gamma} \cos(\frac{\gamma\pi}{2\alpha})} \cdot M \leq M$

$\exists R_0 > 0$ s.t. when $r > R_0$, $|g(z)| \leq M$,

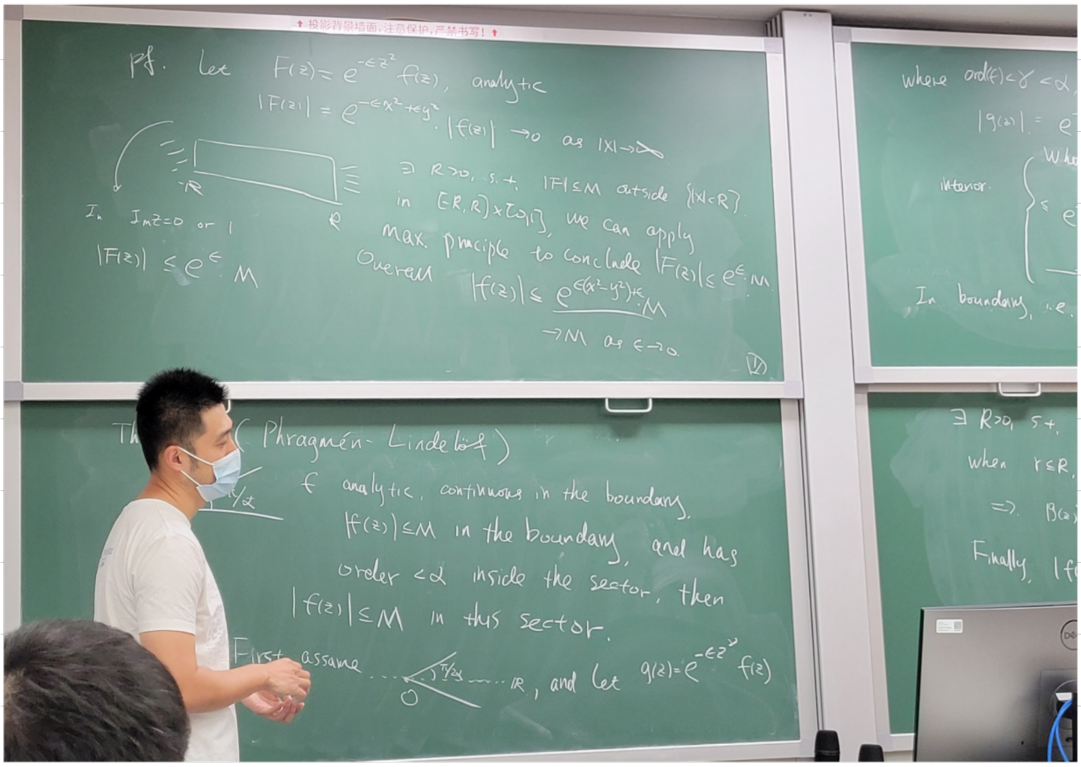
When $r \leq R_0$, $|g(z)| \leq M$ on the boundary of



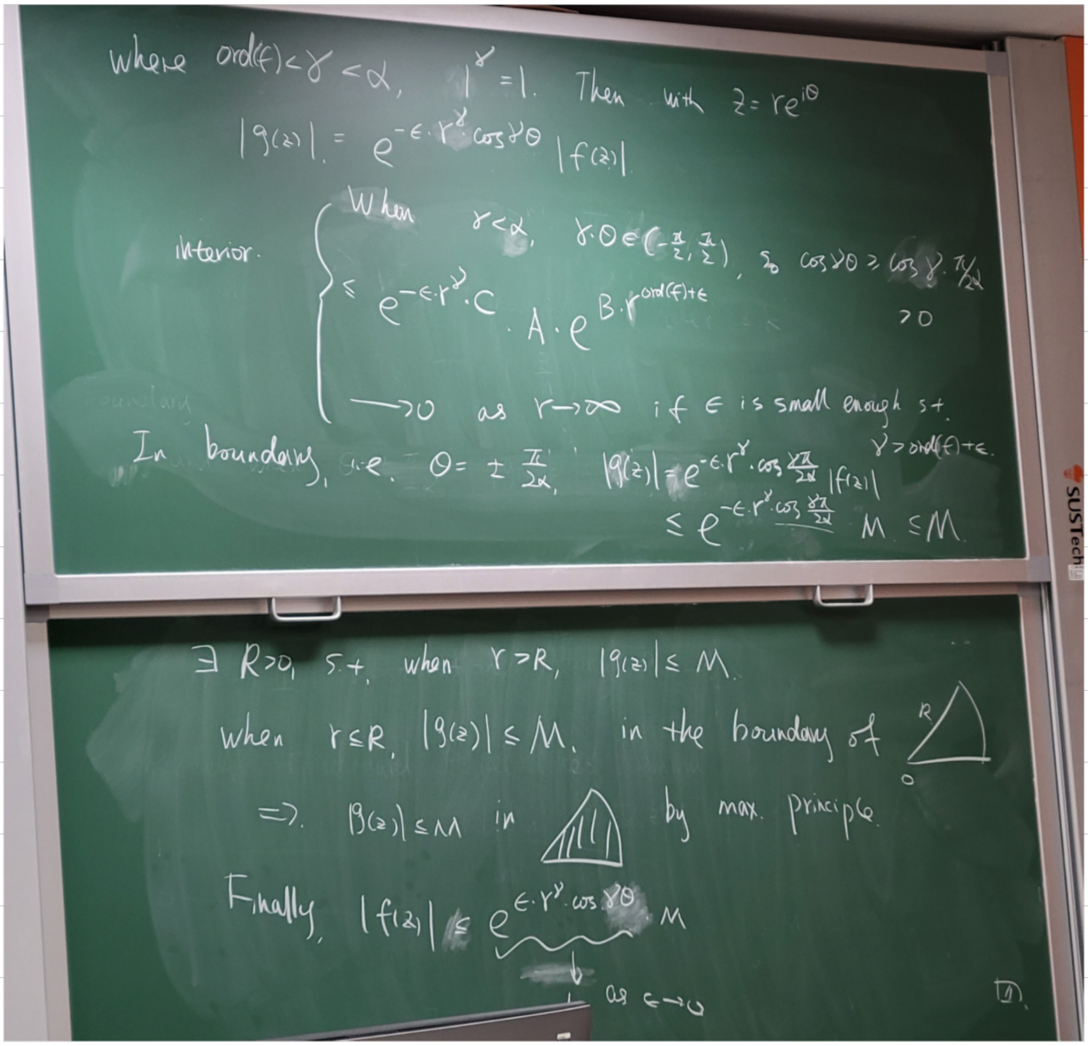
$\Rightarrow |g(z)| \leq M$ in

by maximum principle.

Finally, $|f(z)| \leq e^{\varepsilon r^{\gamma} \cos(\gamma\theta)} \cdot M \longrightarrow 1$, as $\varepsilon \to 0$ ▥



Pf. Let $F(z)=e^{-\epsilon z^2}f(z)$, analytic
$|F(z)|=e^{-\epsilon x^2+\epsilon y^2}\cdot|f(z)|\to 0$ as $|x|\to\infty$
$\exists R>0$, s.t. $|F|\le M$ outside $\{|x|<R\}$
in $[-R,R]\times[0,1]$, we can apply
max. principle to conclude $|F(z)|\le e^{\epsilon}M$
In $\operatorname{Im}z=0$ or 1
$|F(z)|\le e^{\epsilon}M$
Overall $|f(z)|\le e^{\epsilon(x^2-y^2)+\epsilon}M$
$\to M$ as $\epsilon\to 0$
Thm (Phragmén-Lindelöf)
π/α
f analytic, continuous in the boundary,
$|f(z)|\le M$ in the boundary, and has
order $<\alpha$ inside the sector, then
$|f(z)|\le M$ in this sector.
First assume 0, $\pi/2\alpha$, $\mathbb{R}$, and let $g(z)=e^{-\epsilon z^{\gamma}}f(z)$



where $\operatorname{ord}(f)<\gamma<\alpha$, $1^{\gamma}=1$. Then with $z=re^{i\theta}$
$|g(z)|=e^{-\epsilon\cdot r^{\gamma}\cos\gamma\theta}|f(z)|$
When $\gamma<\alpha$, $\gamma\cdot\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, so $\cos\gamma\theta\ge\cos\gamma\cdot\pi/2\alpha$
interior
$\le e^{-\epsilon\cdot r^{\gamma}\cdot C}\cdot A\cdot e^{B\cdot r^{\operatorname{ord}(f)+\epsilon}}$
>0
$\longrightarrow 0$ as $r\to\infty$ if ϵ is small enough s.t.
$\gamma>\operatorname{ord}(f)+\epsilon$.
In boundary, i.e. $\theta=\pm\frac{\pi}{2\alpha}$, $|g(z)|=e^{-\epsilon\cdot r^{\gamma}\cdot\cos\frac{\gamma\pi}{2\alpha}}|f(z)|$
$\le e^{-\epsilon\cdot r^{\gamma}\cdot\cos\frac{\gamma\pi}{2\alpha}}M\le M$
$\exists R>0$, s.t. when $r>R$, $|g(z)|\le M$.
when $r\le R$, $|g(z)|\le M$, in the boundary of
R
0
$\Rightarrow$ $|g(z)|\le M$ in by max. principle
Finally, $|f(z)|\le e^{\epsilon\cdot r^{\gamma}\cos\gamma\theta}M$
as $\epsilon\to 0$

Lecture 6: cont. Entire function of exponential type

Lecture 6-2023 年 3 月 9 日天气晴好 ☼，有点热，18°C-27°C

主要内容：Carleman's Formula, Plancherel-Pólya Theorem

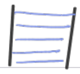
- Torsten Carlmen 瑞典数学家（1892-1949），以经典分析及其应用的成果而闻名，是瑞典最有影响力的数学家。
- 若 Entire function 为 Exponential Type，则可以把它在 x-轴上的信息向上平移。
- Plancherel-Pólya Theorem 是一个 L^p 结果，我们之后会看到 L^2 结果（**更关心**）

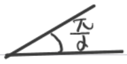
Montel Theorem，常常称 Montel 定理为紧性原理

其他信息：

- sub-harmonic，次调和函数也有极大值原理
- Bernstein inequality

cont. Complex Analysis

① Hadamard three-line lemma 

② Phragmén-Lindelöf  $\text{ord}(f) < \alpha$. cannot be "=" in general, e.g. $e^{z^{\alpha}}$

maximum principle

Cor If $\text{ord}(f) < 1$, bounded in a line, then $f = \text{constant}$

pf: ~ bounded in both sides of this line, then by Liouville thm. ▥

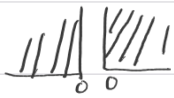
实轴上的信息、 传到 $x+iy$

Thm 11: f entire of exponential type i.e. $|f(z)| \le A \cdot e^{B|z|}$, then $\boxed{\sup_{x \in \mathbb{R}} |f(x)| \le M} \Rightarrow |f(x+iy)| \le e^{B|y|} \cdot M$.

for $\forall y \in \mathbb{R}, x \in \mathbb{R}$

proof: Assume $y>0$, let $g(z) \overset{def}{=} e^{i(B+\varepsilon)z} f(z)$, then $|g(x)| = |f(x)| \le M$. and $|g(iy)| = |e^{-(B+\varepsilon)y} \cdot f(y)|$

$\le e^{-\varepsilon y} \to 0$, as $y \to +\infty$

so $N \overset{def}{=} \sup_{y>0} |g(iy)|$ can be attained. Apply thm 10. to , then we have

$\sup_{Im(z) \ge 0} |g(z)| \le \max\{M, N\}$.

Notice that N cannot be larger than M. so $\sup_{Im z \ge 0} |g(z)| \le M$

$\Rightarrow |f(x+iy)| \le |e^{iBz}| \cdot |g(z)| \le M \cdot e^{B|y|}$. ▥

Remark:

① By thm 11, $B_\tau \overset{def}{=} \{f \text{ entire of exponential type } \tau\}$, is a Banach space under $\|f\| \overset{def}{=} \sup_{x \in \mathbb{R}} |f(x)|$

$\boxed{\text{Bernstein's inequality}}$. $\forall f \in B_\tau$, $\|f'\| \le \tau \|f\|$, and "=" holds if and only if $f = \alpha \cdot e^{i\tau z} + \beta \cdot e^{-i\tau z}$,

$\alpha, \beta \in \mathbb{C}$

see

Exercise 12, 13, not strictly required. (Not in exam)

Thm 12: If f is entire of exponential type. and $\lim_{|x| \to \infty} |f(x)| = 0$, then $\lim_{|x| \to \infty} |f(x+iy)| = 0$, uniform in y in every bounded set.

proof: Recall $\boxed{\text{montel's thm}}$ (thm 3.3, chapter 8, stein) not covered in UG complex analysis

not given in textbook Suppose. $\mathcal{F} = \{f_\alpha\}_\alpha$ is a family of holomorphic functions on Ω, that is uniformly bounded in every compact subset of Ω, then

(1) $\mathcal{F}$ is equi-continuous in every cpt subset of Ω

$\forall \varepsilon > 0, \exists \delta > 0$, s.t. $|f_\alpha(z_1) - f_\alpha(z_2)| < \varepsilon$, $\forall \alpha$, $\forall |z_1 - z_2| < \delta$

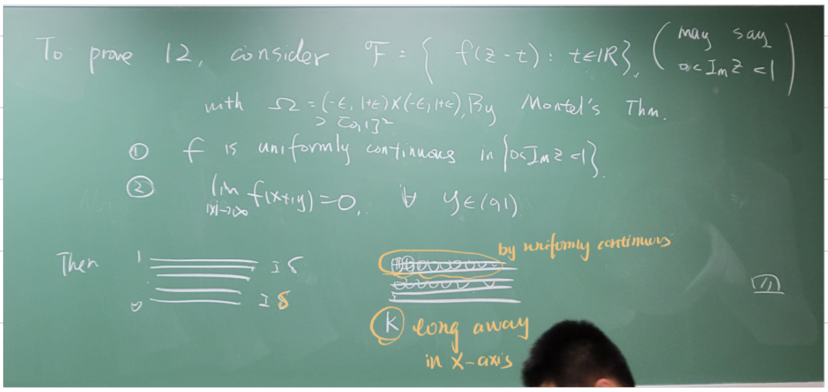
(2) $\mathcal{F}$ is a normal family ($\forall$ sequence in $\mathcal{F}$, $\exists$ subsequence s.t. uniformly convergent in every compact subset)

now, to prove thm 12, consider $\mathcal{F}=\{f(z-t): t\in\mathbb{R}\}$ (may say $0<\mathrm{Im}\, z<1$), with

$\Omega=(-\varepsilon,1+\varepsilon)\times(-\varepsilon,1+\varepsilon)\supseteq[0,1]^2$, now by montel's thm

① f is uniformly continuous in $\{0<\mathrm{Im}\, z<1\}$

② $\lim_{|x|\to\infty} f(x+iy)=0$, $\forall y\in(0,1)$



Exercise 2,3,4

meaning about proof of the 3-line lemma

Introduce a concept: Exponential type 0: $\forall\varepsilon>0, \exists A_\varepsilon>0$ s.t. $|f(z)|\le A_\varepsilon e^{\varepsilon|z|}$.

Final remark: 若为 exponential type, 则可把 X-轴上信息 向上平移

Section 2: Carleman's Formula

Thm 13: Let f be analytic (and no zero in $\partial\Omega$) in $\{\mathrm{Im}\, z\ge 0\}$, $z_k=r_ke^{i\theta_k}$, $k=1,2,\cdots,n$ be its zeros in

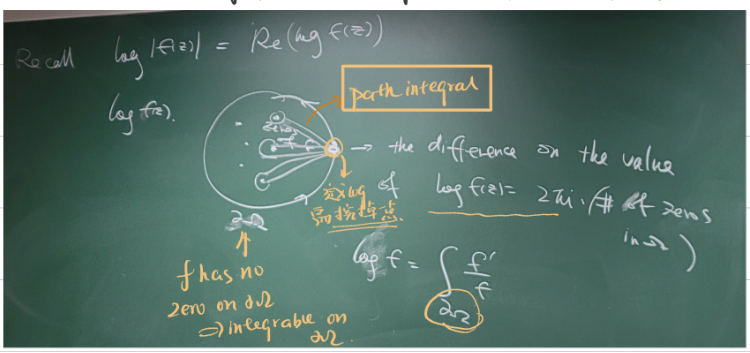
Similar thing like Jensen's formula.

$\Omega \overset{\text{def}}{=}$  In Jensen's formula, we integrate on entire disk, hence Jensen's formula lacks information on X-axis, in thm 13, semi-disk ⇒ Information on X-axis

then
$$\sum_{k=1}^{n}\left(\frac{1}{r_k}-\frac{r_k}{R^2}\right)\sin\theta_k=\frac{1}{\pi R}\int_0^\pi\log|f(Re^{i\theta})|\sin\theta\, d\theta$$
$$+\frac{1}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log|f(x)f(-x)|+O_R(1).$$

comprehensive usage of complex analysis

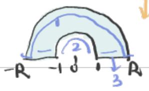
Proof: Recall that $\log|f(z)|=\mathrm{Re}(\log(f(z)))$ (property of log)



end-point of $\partial\Omega$

to prove thm 13. consider $I\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{1}{R^2}-\frac{1}{z^2}\right)\log(f(z))\,dz$

拆分



$$\int_{\frown_{-R}^{R}}+\underbrace{\int_{\frown_{-1}^{1}}}_{O_R(1)}+\underbrace{\int_{-R}^{-1}+\int_1^R}_{=\frac{1}{2\pi i}\left[\int_{-R}^{-1}\left(\frac{1}{R^2}-\frac{1}{x^2}\right)\log f(x)dx+\int_1^R\left(\frac{1}{R^2}-\frac{1}{x^2}\right)\log f(x)dx\right)}$$

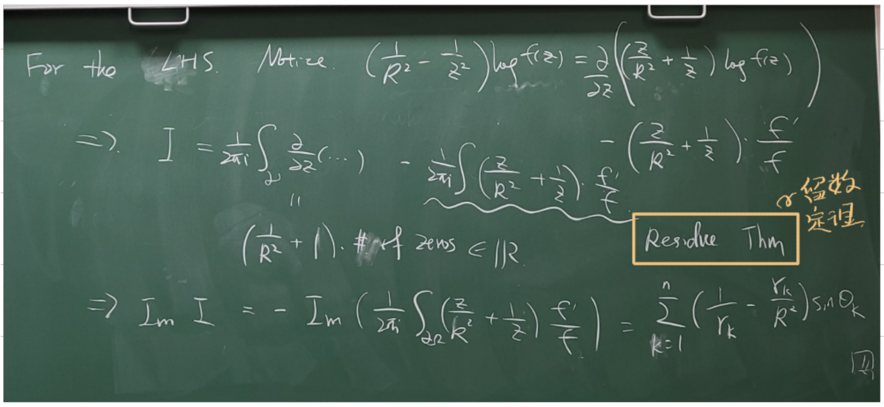
$$\int_{\frown_{-R}^{R}}=\frac{1}{2\pi i}\int_0^\pi\left(\frac{1}{R^2}-\frac{1}{R^2e^{2i\theta}}\right)\log f(z)Re^{i\theta}i\,d\theta$$

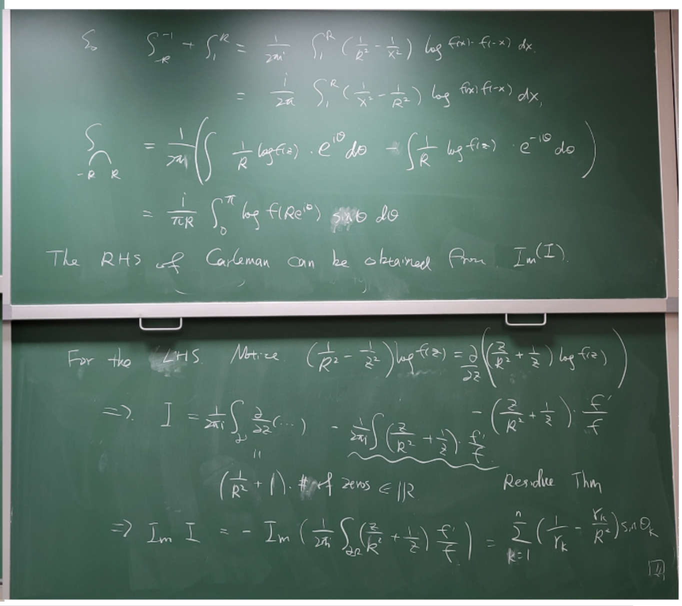
So $\int_{-R}^{-1}+\int_1^R=\frac{1}{2\pi i}\int_1^R\left(\frac{1}{R^2}-\frac{1}{x^2}\right)\log f(x)f(-x)\,dx=\frac{i}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log f(x)f(-x)\,dx.$

and $\int_{\frown_{-R\,R}} = \frac{1}{2\pi i}\left[\int \frac{1}{R}\log f(z) e^{i\theta} d\theta - \int \frac{1}{R}\log f(z)\cdot e^{-i\theta} d\theta\right] = \frac{1}{\pi R}\int_0^{\pi} \log f(Re^{i\theta}) \sin\theta \, d\theta.$

The RHS of Carleman can be obtained from $\operatorname{Im}(I)$ imaginary part

and for the LHS, notice that





Now, we see a corollary of Carleman

Cor (Thm 14): f entire of exponential type, bounded along the real axis, then $\sum \frac{\sin\theta_k}{r_k}$ is absolutely convergent.

$z_k = r_k e^{i\theta_k}$, are zeros of f, $k=1,2,\cdots$

Proof: We may assume f has no zero in the real axis (by continuity argument) might need elaboration, then consider upper/lower half-plane separately, say $\theta_k > 0$, now, by exponential type,

$\log|f(z)| \le C\cdot|z| = C\cdot R$, so the RHS of Carleman is

$$\le \frac{1}{\pi R}\int_0^{\pi} C\cdot R\cdot \sin\theta\, d\theta + \frac{1}{2\pi}\int_1^{R}\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\cdot C\cdot R\cdot dx + O_R(1)$$

当 $R\to\infty$ 为 $O(1)$

$$\le M < \infty, \text{ uniformly in } R$$

Which means $LHS = \sum_{k=1}^{n}\left(\frac{1}{r_k}-\frac{r_k}{R^2}\right)\sin\theta_k < M < \infty$, $\forall R$

$$\sum_{k=1}^{\infty}\left(1-\frac{r_k^2}{R^2}\right)\frac{\sin\theta_k}{r_k}\cdot \chi_{k\le n} \xrightarrow{R\to\infty} \sum_{k=1}^{\infty}\frac{\sin\theta_k}{r_k}$$

we now may go back to basis

辐角

Cor of thm 14 : Thm 15 : If $\{\lambda_n\} \in \mathbb{C}$, $|\arg\lambda_n - \frac{\pi}{2}| \le L < \frac{\pi}{2}$, and $\sum \frac{1}{|\lambda_n|} = \infty$, then $\{e^{i\lambda_n t}\}$ is complete in $C[a,b]$

$\forall -\infty < a < b < +\infty$

Proof : If not, $\exists \mu \in (C[a,b])^*$ s.t. $f(z) = \int_a^b e^{-i\lambda_n t} d\mu(t)$ has zeros $\{\lambda_n\}$

entire function of exponential type

$\Rightarrow \sum \frac{|\sin\theta_k|}{r_k} < \infty$

As $|\arg\lambda_n - \frac{\pi}{2}| \le L < \frac{\pi}{2}$, $|\lambda_k| \approx_L |\operatorname{Im}\lambda_k|$, or $|\sin\theta_k| \approx_L 1$.

$\Rightarrow \sum \frac{1}{r_k} < \infty$, contradiction. $\boxed{///}$

Cor of thm 15 : $0 < \lambda_1 < \lambda_2 < \cdots$ in $\mathbb{R}$, s.t. $\sum \frac{1}{\lambda_n} = \infty$, then $\{t^{\lambda_n}\}$ is complete in $C[a,b]$, $\forall\, 0 < a < b < +\infty$

Proof : consider $\{i\lambda_n\}$, so by thm 15 $\{(e^{-t})^{\lambda_n}\}$ is complete $\Rightarrow \{t^{\lambda_n}\}$ is complete. $\boxed{///}$

Section 2.2.3 Integrability on a line.

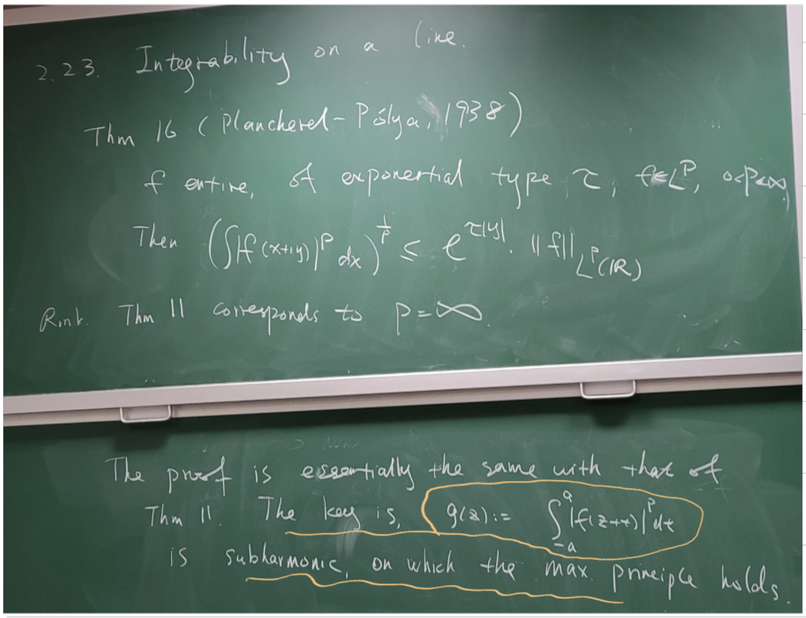
Thm 16 (Plancherel – Pólya, 1938)

later we will see L^2

f entire, of exponential type τ. $f \in L^p$, $0 < p \le \infty$, then $\left(\int |f(x+iy)|^p dx\right)^{\frac{1}{p}} \le e^{\tau|y|} \cdot \|f\|_{L^p(\mathbb{R})}$

Rmk : thm 11 corresponds to $p = \infty$

the proof is essentially the same with that of thm 11



L^2的结果要好看很多:

Lecture 7: Paley-Wiener Theorem and Paley-Wiener space

Lecture 7-2023 年 3 月 14 日星期二 天气晴好 ☼，冷空气，19°C

主要内容：介绍了 Paley-Wiener theorem，并由此重新理解 Paley-Wiener Space

- **Theorem 18:**

其他信息：下周四会讨论这一章节的作业问题，有问题向杨老师汇总。

→ we will finish this chapter this week.

Recap: • $f(z)$ entire functions of exponential type τ, and condition on $f(x)$, $x \in \mathbb{R}$

$$|f(x)| \le M \Rightarrow |f(x+iy)| \le e^{\tau|y|} \cdot M \quad (|f(z)| \le A \cdot e^{\tau|z|})$$

• $|f(x)| \to 0$, as $|x| \to \infty$, then $|f(x+iy)| \to 0$ as $|x| \to \infty$, uniformly in y in every bounded set.

• Carleman's formula 

$$\sum_{k=1}^{n} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \sin\theta_k = \frac{1}{\pi R}\int_0^{\pi} \log|f(Re^{i\theta})| \sin\theta \, d\theta$$

$$+ \frac{1}{2\pi}\int_1^{R} \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log|f(x)f(-x)| \, dx + O(1)$$

• Relatively useful Corollary: $\sum \frac{\sin\theta_k}{r_k}$ is absolutely convergent

• Remark. It is known that $\sum \frac{1}{r_k} = \infty$.

A similar result as the first estimate

Thm 16: $\left(\int_{-\infty}^{+\infty} |f(x+iy)|^p dx\right)^{\frac{1}{p}} \le e^{\tau|y|} \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}}$

Pf: Here we maximum principle of sub-harmonic function! The Rest are almost the same.

Skip the proof.

Thm 17: f is entire of exponential type, $f \in L^p(\mathbb{R})$, for some $0<p<\infty$, then $\forall \varepsilon>0$, $\exists B>0$ s.t.

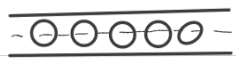
for all increasing sequence $\lambda_1 < \lambda_2 < \cdots$, $|\lambda_{k+1} - \lambda_k| \ge \varepsilon > 0$, we have

$$\sum_n |f(\lambda_n)|^p \le B \int_{-\infty}^{+\infty} |f(x)|^p dx$$

Proof: Since $|f|^p$ is sub harmonic, we have

$$|f(z_0)|^p \le \frac{1}{\pi\delta^2}\iint_{|z-z_0|\le\delta} |f(z)|^p dx\, dy.$$

Take $\delta = \frac{\varepsilon}{2}$, so $\{|z-\lambda_n|<\delta\}$ are disjoint (by $|\lambda_{k+1}-\lambda_k| \ge \varepsilon > 0$), then

$$\sum |f(\lambda_n)|^p \le \frac{1}{\pi\delta^2} \cdot \iint_{|\operatorname{Im} z| \le \delta} |f(z)|^p$$ 

$$= \frac{1}{\pi\delta^2}\int_{-\delta}^{+\delta}\int_{-\infty}^{+\infty} |f(x+iy)|^p dx\,dy \le \underbrace{\boxed{\frac{2 \cdot e^{\tau \cdot |\delta| \cdot p}}{\pi\delta^2}}}_{\text{constant}} \cdot \|f\|_{L^p(\mathbb{R})}^p \quad \blacksquare$$

Exercise 7 is of similar principle: $\|f'\|_{L^p(\mathbb{R})} \le B \cdot \|f\|_{L^p}$

Include derivative, may use thing like Cauchy integral formula.

2.4.4. The Paley-Wiener Thm

• $p=2$ is special, Plancherel, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

• If $\phi(t) \in L^2[-A, A]$, then $f(z) = \int_{-A}^{A} \phi(t) e^{izt} dt$ is entire function of exponential type A, and $f \in L^2(\mathbb{R})$

Thm 18: Let $f(z)$ be an entire function s.t.

$$|f(z)| \le C \cdot e^{A \cdot |z|}, \text{ and } \int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$$

then $\exists \phi \in L^2[-A, A]$ s.t. $f(z) = \int_{-A}^{A} \phi(t) e^{izt} dt$

Stein has given a special in his complex analysis (without using real analysis)

Cor: $|f(z)|\cdot e^{-A|y|} \to 0$, as $|z|\to\infty$

proof: since $f(z)=\int_{-A}^{A}\phi(t)e^{izt}dt$, $\forall\varepsilon>0$, $\exists\psi\in C_0^\infty[-A,A]$, s.t. $\|\phi-\psi\|_{L^1}<\varepsilon$

$$= \int_{-A}^{A}\psi(t)e^{izt}dt + \underbrace{\int_{-A}^{A}(\phi-\psi)e^{izt}dt}_{\le e^{A|y|}\cdot\varepsilon}$$

$$\le C_N\cdot\frac{e^{A|y|}}{(1+|z|)^N}$$

$\Rightarrow \lim_{|z|\to\infty}|f(z)|\cdot e^{-A|y|}\le\varepsilon$, $\forall\varepsilon>0$. $\square$

Example: If $f\in B_\tau$, entire of exponential type τ, $\sup_{x\in\mathbb{R}}|f(x)|<\infty$,

then $f(z)=f(0)+z\int_{-\tau}^{\tau}\phi(t)e^{izt}dt$, $\phi\in L^2$

proof: $\frac{f(z)-f(0)}{z}\in L^2(\mathbb{R})$, then apply thm 18.

Example (Bernstein inequality) $\|f'\|_{L^\infty(\mathbb{R})}\le\tau\cdot\|f\|_{L^\infty(\mathbb{R})}$

proof: Consider $g_\varepsilon(z)=\underbrace{\underbrace{f(z)}_{\text{exponential type }\tau}\cdot\underbrace{\frac{\sin(\varepsilon z)}{\varepsilon z}}_{\text{exponential type }\varepsilon}}_{\text{exponential type }\varepsilon+\tau}\in L^2(\mathbb{R})$

Now by Paley-Wiener thm, $\exists\phi_\varepsilon$ s.t. $g_\varepsilon(z)=\int_{-\tau-\varepsilon}^{\tau+\varepsilon}\phi_\varepsilon(t)e^{izt}dt$

Since Bernstein can be proved for this type (Problem 12, last week), then $\|g_\varepsilon'\|_{L^\infty}\le(\tau+\varepsilon)\|g_\varepsilon\|_{L^\infty}$,

done by $\varepsilon\to0$. $\square$

Example: $f(z)$ entire exponential type $\tau<2\pi$, $\int_{-\infty}^{\infty}|f(x)|dx<\infty$, then

$$\sum_{n=-\infty}^{\infty}f(n)=\int_{-\infty}^{+\infty}f(x)dx$$

see above prob

proof: $f\in L^1\Rightarrow f'\in L^1\Rightarrow f$ is B.V. (Bounded Variation)

Poisson summation formula

$\sum f(n)=\sum\hat{f}(n)$, where $\hat{f}(\xi)\overset{\text{def}}{=}\int_{-\infty}^{\infty}f(x)e^{-2\pi ix\cdot\xi}dx$

But since $|f(x)|\to0$ as $|x|\to\infty$, $f\in L^1(\mathbb{R})\Rightarrow f\in L^2(\mathbb{R})$. by Paley-Wiener, $f(z)=\int_{-\tau}^{\tau}\phi(t)e^{izt}dt$

$\Rightarrow\hat{f}=0$ outside $[-\frac{\tau}{2\pi},\frac{\tau}{2\pi}]\subsetneq(-1,1)\Rightarrow\hat{f}(n)=0$, $n\ne0$

$\Rightarrow\sum f(n)=\hat{f}(0)=\int_{-\infty}^{+\infty}f(x)dx$. $\square$

Now we go back to the proof of thm 18 (Paley-Wiener)

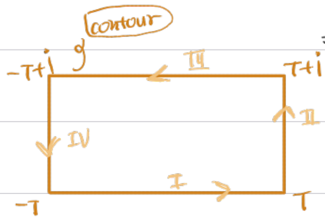
proof: Since $f\in L^2(\mathbb{R})$, $\phi(t)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-ixt}dx$ is well-defined L^2-function. It suffices to prove

supp $\phi\subseteq[-A,A]$, If so, by Fourier Inversion

$f(x)=\int_{-A}^{A}\phi(t)e^{itx}dt$ that extends to $\mathbb{C}$ $\Rightarrow f(z)=\int_{-A}^{A}\phi(t)e^{izt}dt$

Now, say $t<-A$, $\phi(t)=\int_{-\infty}^{\infty}f(x)e^{-ixt}dx$

contour



$=\lim_{T\to\infty}\int_{-T}^{T}f(x)e^{-ixt}dx$

$0=\int_{\square}f(z)e^{izt}dz$

We need to show that $I\to 0$ as $T\to\infty$

For III $\left|\int_{-T}^{T}f(x+iT)e^{-i(x+iT)t}dx\right|\le e^{Tt}\int_{-T}^{T}|f(x+iT)|dx$

Cauchy-Schwarz

$\le e^{Tt}\sqrt{2T}\left(\int_{-\infty}^{+\infty}|f(x+iT)|^2dx\right)^{\frac{1}{2}}$

$\le e^{AT}\|f\|_{L^2(\mathbb{R})}$

$\to 0$, as $T\to\infty$, since $t<-A$

For II

$\int_0^T f(T+ix)e^{-i(T+ix)t}dx$, $\forall\varepsilon>0$, $\exists R$ s.t. $e^{(A+t)R}<\varepsilon$

$\int_0^T=\int_0^R+\int_R^{\infty}$

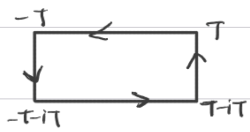
$\int_0^R$: $\le e^{Rt}\int_0^R|f(T+ix)|dx$ $\to 0$, as $T\to\infty$.

$\int_R^{\infty}$: $\le\int_R^T|f(T+ix)|e^{xt}dx$, where $|f(T+ix)|\le e^{Ax}\|f\|_{\infty(\mathbb{R})}$

$\le\|f\|_{\infty(\mathbb{R})}\cdot e^{(A+t)R}$

$\le\varepsilon\cdot\|f\|_{\infty(\mathbb{R})}$

IV is the same as II, Hence $\boxed{\phi(t)=0,\ \forall t<-A}$. Now for $t>A$ consider



$\boxed{///}$

2.2.5 The Paley-Wiener Space $PW_{[-\pi,\pi]}$

Recall $PW_{[-\pi,\pi]}\overset{\text{def}}{=}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)e^{izt}dt,\ \phi\in L^2[-\pi,\pi]\right\}$

By Paley-Wiener thm ① $=\{f$ entire, exponential type (π), $f\in L^2(\mathbb{R})\}$

$\langle f,g\rangle_{PW}=\langle\phi_f,\phi_g\rangle_{L^2[-\pi,\pi]}$

② • Notice the convergence in PW implies uniform convergence.

$|f(x+iy)|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)e^{izt}dt\right|$

$\le c\,e^{|y|\cdot\pi}\cdot\underbrace{\|\phi\|_{L^2[-\pi,\pi]}}_{=\|f\|_{PW}}$

③ Since $\{e^{int}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi,\pi]$, then

$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}\cdot e^{izt}dt=\frac{\sin\pi(z-n)}{\pi(z-n)}$ is an orthonormal basis for $PW_{[-\pi,\pi]}$, therefore

$f(z)=\sum_{n=-\infty}^{\infty}c_n\cdot\frac{\sin\pi(z-n)}{\pi(z-n)}$

To complete c_n. $c_n=\left\langle f,\frac{\sin\pi(z-n)}{\pi(z-n)}\right\rangle$

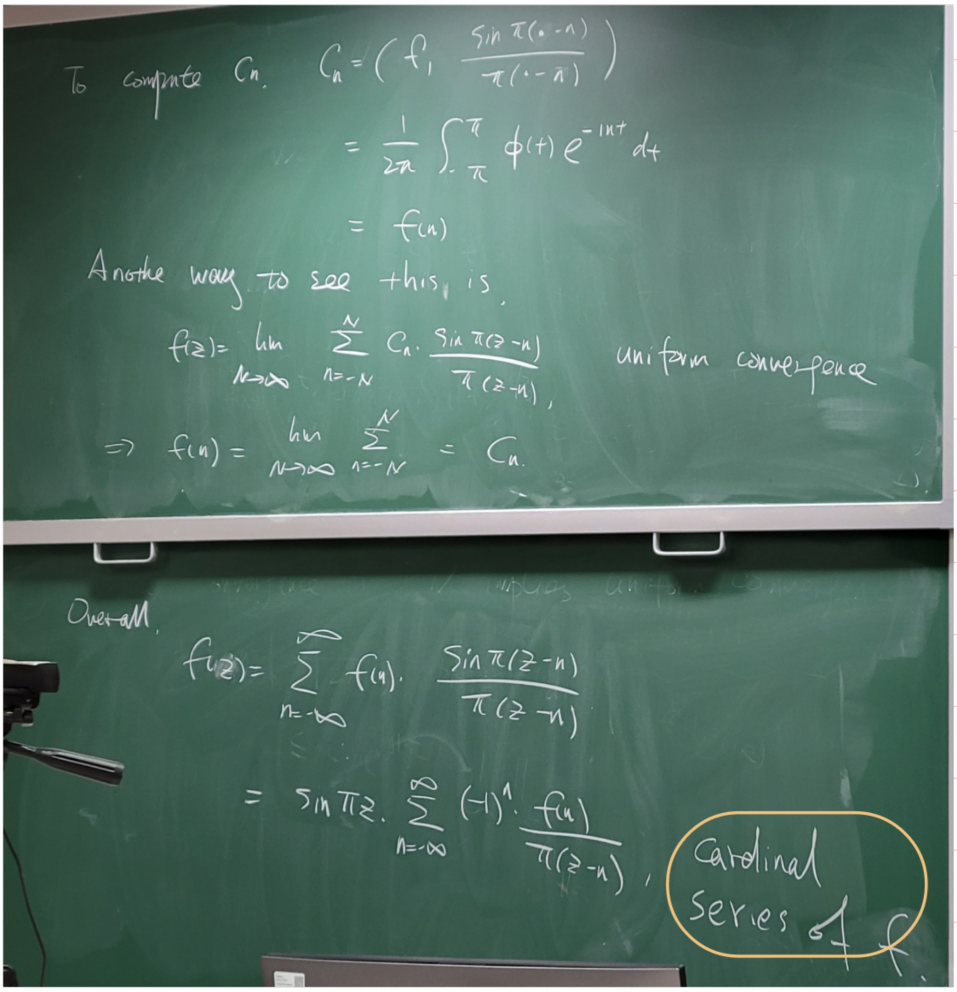
$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)e^{-int}dt=f(n)$

Another way to see this is

$f(z) = \lim_{N\to\infty} \sum_{n=-N}^{N} C_n \frac{\sin\pi(z-n)}{\pi(z-n)}$ uniform convergence

$\Rightarrow f(n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \; = C_n$

Overall. $f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin\pi(z-n)}{\pi(z-n)}$

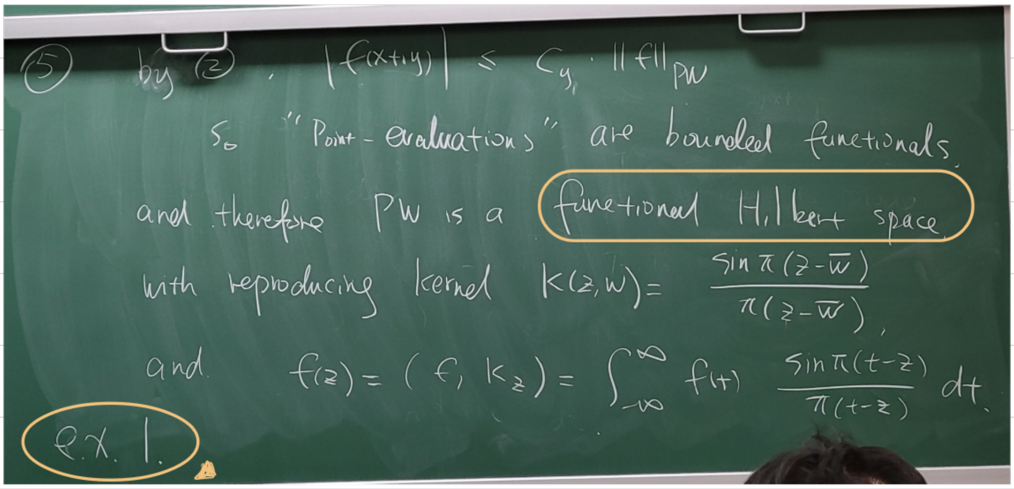


④ $f \in PW \Rightarrow f' \in PW$

$f' = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \cdot it \cdot e^{izt} dt$

$\|f'\|_{PW} = \|t \cdot \phi(t)\|_{L^2[-\pi,\pi]} \le \pi \cdot \|\phi\|_{L^2} = \pi \cdot \|f\|_{PW}$

⑤



Next week thursday. about HW for chapter 2

Lecture 8: The Completeness of sets of complex exponentials

Lecture 8-2023 年 3 月 16 日星期四 天气晴好 ☼，升温，20°C

主要内容：进入第三章，The Completeness of sets of complex exponentials

- **Exact** sequence
- 继续研究第一章中介绍过的 Kadec $\frac{1}{4}$-theorem

其他信息：

Chapter 3: The completeness of sets of complex exponentials

We may assume $\{\lambda_n\}$ are distinct. if some λ has multiplicity m, then every result in this chapter still holds with $\{\cdots, e^{i\lambda t}, t\cdot e^{i\lambda t}, \cdots, t^{m-1}e^{i\lambda t}, \cdots\}$

In this chapter, we may consider entire function of $\boxed{\text{Fourier transform}}$ form.

$$f(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt} dt$$

Recall that $\{e^{int}\}_{n\in\mathbb{Z}}$ is complete in $L^p[-\pi,\pi]$, $1\le p<\infty$.

Def: A sequence is called exact, if it is complete, but fails to be complete by removing any term.

If it becomes exact when N to be removed (added), we say it has excess (deficiency) N.

周期的 ⇒ 构假也为周期的　　　需加入一个有效的

Proposition: $\{e^{int}\}$ is exact in $L^p[-\pi,\pi]$, $1\le p<\infty$, but has deficiency 1 in $C[-\pi,\pi]$

proof: If we remove any e^{in_0t}, then $e^{in_0t}\in L^{p'}\setminus\{0\}$, while $\int_{-\pi}^{\pi} e^{int}\cdot e^{-in_0t}dt=0$, $\forall n\ne n_0$

$\Rightarrow \{e^{int}\}_{n\in\mathbb{Z}\setminus\{n_0\}}$ is not complete. (由 complete 定义)

In $C[-\pi,\pi]$, one can see that $\{e^{int}\}_{n\in\mathbb{Z}}$ is not complete. Because in general, we do not have $f(-\pi)=f(\pi)$. To make it complete add $e^{i\mu t}$, $\mu\in\mathbb{R}$, s.t. $e^{i\pi\mu}\ne e^{-i\pi\mu}$.

Then $\forall f\in C[-\pi,\pi]$,

consider $F(t)=f(t)-c\cdot e^{i\mu t}$, where $F(\pi)=F(-\pi)$

$$\Downarrow$$
$$c=\frac{f(\pi)-f(-\pi)}{e^{i\pi t}-e^{-i\pi t}}$$

then F can be approximated by $\{e^{int}\}_{n\in\mathbb{Z}}$

///

Proposition: $\{e^{int}\}_{n\in\mathbb{Z}}$ is in-complete in every $L^p[-\pi-\varepsilon,\pi+\varepsilon]$

proof: Consider $\phi(t)=\begin{cases}-1, & -\pi-\varepsilon\le t\le-\pi+\varepsilon\\ 0, & \text{otherwise}\\ 1, & \pi-\varepsilon\le t\le\pi+\varepsilon\end{cases}$, then

构造一个

$$\int_{-\pi-\varepsilon}^{\pi+\varepsilon}\phi(t)e^{int} = -\int_{-\pi-\varepsilon}^{-\pi+\varepsilon}e^{int}dt+\int_{\pi-\varepsilon}^{\pi+\varepsilon}e^{int}dt = -\int_{\pi-\varepsilon}^{\pi+\varepsilon}e^{-int}dt+\int_{\pi-\varepsilon}^{\pi+\varepsilon}e^{int}dt=(2i)\int_{\pi-\varepsilon}^{\pi+\varepsilon}\sin(nt)dt$$

$=0$, but $\phi\ne0$

Hence $\{e^{int}\}_{n\in\mathbb{Z}}$ is not complete (by def)

Now, we remove $n\le0$, to obtain $\{e^{int}\}_{n>0}$

Thm 1 (Careleman, 1922)

Suppose $\lambda_n>0$, and $\limsup_{R\to\infty}\left(\frac{1}{\log(R)}\sum_{\lambda_n<R}\frac{1}{\lambda_n}\right)>\frac{A}{\pi}$, then $\{e^{i\lambda_n t}\}$ is complete in $C[-A,A]$.

Corollary: $\{e^{int}\}_{n=1}^{\infty}$ is complete in $C[-A,A]$
$L^p[-A,A],\ 1\le p<\infty$

proof: $\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\to 1$

Remark: Removing finitely many terms does not change anything

proof of thm 1: If not, $\exists$ B.V. (bounded variation) function $w\neq 0$ s.t.

$$f(z)=\int_{-A}^{A}e^{izt}\,dw(t) \text{ vanishes at } \lambda_n$$

Recall the Carleman's formula to the right half-plane ($f(iz)$) 转一下

$$\sum_{r_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\le \frac{1}{\pi R}\int_0^{\pi}\left|\log|f(Re^{i\theta})|\right|\sin\theta\,d\theta$$
λ_k 纯为zeros

$O(1)$, as $|f(re^{i\theta})|=O(e^{Ar|\sin\theta|})$

$$+\frac{1}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log|f(x)f(-x)|\,dx+O(1)$$

$\le 2A\cdot|x|+O(1)$

Now

$$\sum_{r_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\le 2A\cdot\frac{1}{2\pi}\int_1^R\left(\frac{x}{x^2}-\frac{x}{R^2}\right)dx+O(1)$$

$=\frac{1}{2\pi}\log R+O(1)$

$$\le\frac{A}{\pi}\log(R)+O(1)$$

相除

therefore $\limsup\limits_{R\to\infty}\left[\frac{1}{\log(R)}\sum_{r_n<R}\left(\frac{1}{\lambda_n}-\frac{\lambda_n}{R^2}\right)\right]\le\frac{A}{\pi}$

It remains to show that $\limsup_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$ (= the above)

Since $\lambda_n\ge 0$, then $\limsup\limits_{R\to\infty}\left[\frac{1}{\log(R)}\sum_{r_n<R}\left(\frac{1}{\lambda_n}-\frac{\lambda_n}{R^2}\right)\right]\le\limsup\limits_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$

Now for $\forall\beta\in(0,1)$, $\sum_{\lambda_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\ge\sum_{\lambda_k<\beta R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)$

$$=\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}\left(1-\frac{\lambda_k^2}{R^2}\right)$$

$$\ge(1-\beta^2)\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}$$

$\Rightarrow\frac{1}{\log(R)}\sum_{\lambda_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\ge(1-\beta^2)\frac{1}{\log(R)}\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}$, take $\limsup_{R\to\infty}$

then $\limsup_{R\to\infty}(1-\beta^2)\frac{1}{\log(R)}\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}=(1-\beta^2)\limsup\limits_{R\to\infty}\frac{1}{\log(\beta R)}\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}$

$=(1-\beta^2)\limsup\limits_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$, $\forall\,0<\beta<1$

then take $\beta\to 0$, $\limsup_{R\to\infty}\text{LHS}\ge\limsup\limits_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$

$\frac{A}{\pi}$ (= LHS)

Contradiction to our assumption! ///

Exercise (弱条件): Thm If $\lambda_n>0$, $\liminf\limits_{n\to\infty}\frac{n}{\lambda_n}>\frac{A}{\pi}$, then $\{e^{i\lambda_n t}\}$ is complete in $C[-A,A]$

We may reduce it to above argument

Remark: In fact, $\limsup\limits_{n\to\infty}\frac{n}{\lambda_n}>\frac{A}{\pi}$ is enough.

3.2 Exponentials close to the trigonometric system.

Recall Kadec's $\frac{1}{4}$ - thm: $\lambda_n \in \mathbb{R}$

$\{e^{i\lambda_n t}\}_n$ is a Riesz basis for $L^2[-\pi,\pi]$, if $|\lambda_n - n| \le L < \frac{1}{4}$

proof relies on certain expansion

Thm 3: Given $\{\lambda_n\} \subseteq \mathbb{C}$, denote $n(r) \overset{\text{def}}{=} \#\{|\lambda_n| \le r\}$, and $N(r) = \int_1^r \frac{n(t)}{t} dt$

then $\{e^{i\lambda_n t}\}$ is $\boxed{\text{complete}}$ in $L^p[-\pi,\pi]$, $1<p<\infty$, if

$\varliminf_{r\to\infty} (N(r) - 2r + \frac{1}{p}\log(r)) > -\infty$

Remark: If $\{\lambda_n\}$ is complete then $\forall \lambda \in \mathbb{C}$, $\{\lambda_n - \lambda\}$ is also complete.

平移不改变完备性 $\int_{-\pi}^{\pi} \phi(t) e^{i(\lambda_n - \lambda)t} dt = \int_{-\pi}^{\pi} (\phi(t)e^{-i\lambda t}) e^{i\lambda_n t} dt$

proof of thm3: If not, $\exists \phi \in L^{p'}$ s.t. $f(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt} dt$, vanishes at $\{\lambda_n\}$, we may assume

$\|\phi\|_{L^{p'}} = 1$

Recall that Jensen's formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| d\theta - \log|f(0)| = \sum_{k=1}^{n} \log\left(\frac{r}{r_k}\right) = \boxed{\int_0^r \frac{n(t)}{t} dt}$$

$$= N(r) + O(1)$$

We now write $f(z) = \underbrace{\int_{-\pi+\varepsilon}^{\pi-\varepsilon}}\ + \int_{\pi-\varepsilon \le |t| \le \pi}$ exponential decay (ε)

now by Holder $\le \underbrace{\|\phi\|_{L^{p'}}}_{=1} \left(\int_{-\pi+\varepsilon}^{\boxed{\pi-\varepsilon}} e^{|y|\cdot t\cdot p} dt\right)^{\frac{1}{p}} + \boxed{O_\varepsilon(1)} \left(\int_{-\pi}^{\pi} e^{|y|\cdot t \cdot p} dt\right)^{\frac{1}{p}}$

$\pi-\varepsilon$ 引入exponential decay(ε)

小量, extra decay

$$\le C\left[e^{(\pi-\varepsilon)|y|} \cdot |y|^{-\frac{1}{p}} + O_\varepsilon(1) \cdot e^{\pi|y|} \cdot |y|^{-\frac{1}{p}}\right)$$

$$= C \cdot e^{\pi|y|} |y|^{-\frac{1}{p}} \left[e^{-\varepsilon|y|} + O_\varepsilon(1)\right]$$

Now substitute the above estimate of f into Jensen's formula

$LHS = N(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{\pi|y|} \cdot |y|^{-\frac{1}{p}} \cdot (e^{-\varepsilon|y|} + O_\varepsilon(1))\right| d\theta$, here $y = r\sin\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \pi\cdot r\cdot|\sin\theta| - \underline{\frac{1}{p}\log(r) - \frac{1}{p}\log|\sin\theta|} + \log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)| \, d\theta$$

大O吸收

$$= 2r - \frac{1}{p}\log(r) + O(1) + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)| d\theta$$

$$\Rightarrow N(r) - 2r + \frac{1}{p}\log(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)| d\theta + O(1)$$

now taking $\limsup_{R\to\infty}$ Note that $\limsup RHS = -\liminf \int -$ Fatou

one can take ε small s.t. $O_\varepsilon(1) < \frac{1}{2}$

then we r is large, $e^{-\varepsilon r|\sin\theta|} < \frac{1}{2}$, $\sin\theta \ne 0$

$\Rightarrow \log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)| < 0$

Hence $\limsup RHS = -\liminf \int -\log|\sim|$

cont. $\le \int \limsup_{R\to\infty} \log\left|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)\right| d\theta$

$\sim \log(0)\, G(1)$ close to $-\infty$. $\boxed{///}$

Direct Corollary

Thm 4: $1<p<\infty$, $\lambda_n \in \mathbb{C}$, $|\lambda_n| \le n + \frac{1}{2p}$, then $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$, the constant $\frac{1}{2p}$ is optimal.

proof: To see the completeness. notice that

$$N(r) = \int_1^r \frac{n(t)}{t}\,dt \geq \int_1^r \frac{1+2[t-\frac{1}{2p}]}{t}\,dt$$ 取整

$$= \int_1^r \frac{2(t-\frac{1}{2p})}{t}\,dt + \int_1^r \frac{1+2[t-\frac{1}{2p}]-2(t-\frac{1}{2p})}{t}$$

$$= 2r - \frac{1}{p}\log(r) + \boxed{\frac{1}{2}\int_1^r \frac{\frac{1}{2}+(t-\frac{1}{2p})-(t-\frac{1}{2p})}{t}\,dt} + O(1)$$

finite

may seen in exercises $\int_1^r \frac{\frac{1}{2}+[x]-x}{x}\,dx$

Hence $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$. $\boxed{///}$

Some arguments come from research papers!

Lecture 9: cont.Completeness of sets of complex exponentials, Kadec $\frac{1}{4}$ theorem

Lecture 9-2023 年 3 月 23 日星期四 天气阴，有点热，23°C-27°C

主要内容： 基本证明了 Kadec $\frac{1}{4}$-theorem

其他信息：

Recap: thm 1: $\lambda_n>0$, and $\limsup_{R\to\infty}\left(\frac{1}{\log(R)}\sum_{\lambda_n<R}\frac{1}{\lambda_n}\right)>\frac{A}{\pi}$, the $\{e^{i\lambda_n t}\}$ is complete in $C[-A,A]$

Rmk (mathematical analysis): $\lambda_n>0$, $\liminf_{n\to\infty}\frac{n}{\lambda_n}>\frac{A}{\pi}$ is sufficient.

In fact $\limsup_{n\to\infty}\frac{n}{\lambda_n}>\frac{A}{\pi}$ is enough!

Thm 3: $\lambda_n\in\mathbb{C}$, $n(r)\overset{def}{=}\#\{\lambda_n:|\lambda_n|\le r\}$, $N(r)=\int_1^r\frac{n(t)}{t}dt$, then $\{e^{i\lambda_n t}\}$ complete in $L^p[-\pi,\pi]$, $1<p<\infty$, if

$\lim_{r\to\infty}\left(N(r)-2r+\frac{1}{p}\log r\right)>-\infty$

Direct Corollary thm 4: $1<p<\infty$, $\lambda_n\in\mathbb{C}$, $|\lambda_n|\le|n|+\boxed{\frac{1}{2p}}$, then $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$

constant $\frac{1}{2p}$ is best possible

quite tricky

cont. proof of the optimality of $\frac{1}{2p}$:

take $\lambda_n=\begin{cases} n+\frac{1}{2p}+\varepsilon, & n>0\\ 0, & n=0\\ n-\frac{1}{2p}-\varepsilon, & n<0\end{cases}$, ε is an arbitrarily positive number.

Consider $\phi(t)=[\cos(\frac{t}{2})]^{\frac{1}{p}-1+2\varepsilon}\cdot\sin\frac{1}{2}t$

$\phi\in L^q[-\pi,\pi]$, as $\left([\cos(\frac{t}{2})]^{-\frac{1}{q}+2\varepsilon}\sin(\frac{1}{2}t)\right)^q$ 可积 有但的

We shall show that $\int_{-\pi}^{\pi}\phi(t)e^{i\lambda_n t}dt=0,\ \forall n$

$\phi(t)$ orthogonal to every element $e^{i\lambda_n t}$

First $\phi\in L^{p'}=L^q=(L^p)^*$

$n=0$, $\lambda_0=0$ (by def), $\int_{-\pi}^{\pi}\phi(t)dt=0$ ✓

$n>0$, since $\sin(\frac{t}{2})=\frac{e^{i\frac{t}{2}}-e^{-i\frac{t}{2}}}{2i}$, $\cos(\frac{t}{2})=\frac{e^{i\frac{t}{2}}+e^{-i\frac{t}{2}}}{2}$, the complex form, then let $c=\frac{1}{2p}+\varepsilon$, $\phi=(\cos\frac{t}{2})^{2c-1}\sin(\frac{t}{2})$

$$\int_{-\pi}^{\pi}\phi(t)e^{i\lambda_n t}=i\cdot 2^{-2c}\int_{-\pi}^{\pi}(1+e^{it})^{2c-1}(1-e^{it})e^{int}dt$$

$$=\lim_{r\to1^-}\int_{-\pi}^{\pi}\underbrace{(1+re^{it})^{2c-1}}\cdot(1-e^{it})e^{int}dt$$

插入r后，可以展开

$=\sum_{k=0}^{\infty}\binom{2c-1}{k}r^k e^{ikt}$

$$=\lim_{r\to1^-}\sum_{k=0}^{\infty}\binom{2c-1}{k}r^k\int_{-\pi}^{\pi}e^{i(n+k)t}(1-e^{it})dt$$

$=0$, for $n=1,2,3,\ldots$

$n<0$ is of similar principle. ∎

Existence of $\phi_n\neq0$ make $\{e^{i\lambda_n t}\}$ not complete $\Rightarrow$ optimality of $\frac{1}{2p}$

Remark: Thm 4 $\boxed{\text{fails}}$ for $p=1$, take $\lambda_n=\begin{cases}n+\frac{1}{2}, & n>0\\ 0, & n=0\\ n-\frac{1}{2}, & n<0\end{cases}$, then $\sin(\frac{t}{2})$ is orthogonal to every $e^{i\lambda_n t}$.

- Thm 4 provides simple examples of sets that are complete in $L^p[-\pi,\pi]$, but fail to be complete in $L^r[-\pi,\pi]$, $r>p$. (对有限测度空间 L^p 的包含关系)

Recall the Kadec-$\frac{1}{4}$-thm

counter-example. (complete but not a basis)

3.3: Counter example $\quad \lambda_n = \begin{cases} n-\frac{1}{4}, & n>0 \\ n+\frac{1}{4}, & n<0 \end{cases}$, no term associated to $n=0$

Thm 5 $\{e^{\pm i(n-\frac{1}{4})t}\}_{n>0}$ is exact in $L^2[-\pi,\pi]$, but not a (Riesz) basis

proof: First show it is complete, to do this we translate $\underbrace{\{\pm(n-\frac{1}{4})\}}_{\{\cdots,-2+\frac{1}{4},-1+\frac{1}{4},1-\frac{1}{4},2-\frac{1}{4},\cdots\}}$ by $\frac{1}{2}$ to

$\{\cdots, -2+\frac{3}{4}, \underbrace{-1+\frac{3}{4}}_{-1-\frac{1}{4}}, \underbrace{1+\frac{1}{4}}_{0-\frac{1}{4}}, 2+\frac{1}{4}, \cdots\}$

$\cdots, \lambda_1, \lambda_0, \lambda_1, \lambda_2, \cdots$

(why translation is allowed: $f \mapsto e^{-i\lambda t} f$ is $\boxed{\text{isomorphism}}$ in $L^2[-\pi,\pi]$, or $C[a,b]$.)

the translated λ_n satisfy $|\lambda_n| \le |n| + \frac{1}{2\cdot 2} = |n| + \frac{1}{4}$, then by thm 4 $\Rightarrow$ completeness!

To see the exactness, fails to be complete on the removal of any one term

Consider $f(z) = \int_{-\pi}^{\pi} [\cos(\frac{t}{2})]^{-\frac{1}{2}} e^{izt} dt$ (bounded in $\mathbb{R}$, not in PW)!

$$\int_{-\pi}^{\pi} (\cos \tfrac{1}{2}t)^{-1/2} e^{i\lambda_n t}\, dt = \sqrt{2} \int_{-\pi}^{\pi} (1+e^{it})^{-1/2} e^{int}\, dt$$
$$= \sqrt{2} \lim_{r\to 1-} \int_{-\pi}^{\pi} (1+re^{it})^{-1/2} e^{int}\, dt$$
$$= \sqrt{2} \lim_{r\to 1-} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} r^k \int_{-\pi}^{\pi} e^{i(n+k)t}\, dt = 0.$$

$\Rightarrow f(0) \neq 0$, $f(\lambda_n) = 0$, $\forall n$

then $\forall n_0$, $\frac{f(z)}{z-\lambda_{n_0}} \in PW$ (remove 掉), non-zero function.

$\| $ by PW

$\int_{-\pi}^{\pi} \phi(t) e^{izt} dt$ for some $\phi \in L^2 \setminus \{0\}$, i.e. ϕ is orthogonal to $\{e^{i\lambda_n t}\}_{n\neq n_0}$

$\Downarrow$

$\{e^{i\lambda_n t}\}$ is exact by definition.

It remains to show that $\{e^{i\lambda_n t}\}$ is not a Riesz basis

Consider $f(z) = \frac{\Gamma^2(\frac{3}{4})}{\Gamma(\frac{3}{4}+z)\cdot\Gamma(\frac{3}{4}-z)}$, where $\Gamma^{-1}(z) = z \cdot e^{\gamma z} \prod_{n=1}^{\infty} (1+\frac{z}{n}) e^{-\frac{z}{n}}$ (γ: Euler constant)

$\Downarrow$

$f(\lambda_n) = 0$

and $f'(\lambda_n) = (-1)^n \cdot \Gamma^2(\frac{3}{4}) \cdot \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}$, $n>0$, and $\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} \sim \frac{1}{\sqrt{n}}$ (omit the calculation)

$\boxed{\text{If } \{e^{i\lambda_n t}\} \text{ is a Riesz basis,}}$ then $1 = \sum c_n e^{i\lambda_n t}$, $\boxed{\sum |c_n|^2 < \infty}$ — 这句用于 Argue by contradiction.

then by taking Fourier transform

$\frac{\sin(\pi z)}{\pi z} = \sum c_n \cdot \underbrace{\frac{\sin(z-\lambda_n)}{\pi(z-\lambda_n)}}_{K_{\lambda_n}(z),\ \text{reproducing kernel.}}$, then

$\langle \underbrace{\frac{f(z)}{f'(\lambda_n)(z-\lambda_n)}}_{\|\text{def}\ \ f_n}, K_{\lambda_m}(z) \rangle = \delta_{n,m} \Rightarrow \{f_n\}, \{K_{\lambda_m}\}$ biorthogonal to each other

$\Rightarrow \forall f$, $f = \sum (f, f_n) K_{\lambda_n}$ (by bi-orthogonality), then

$\frac{\sin(\pi z)}{\pi z} = \sum \underbrace{(\frac{\sin \pi z}{\pi z}, f_n)}_{c_n = f_n(0) = \frac{f(0)}{\lambda_n f'(\lambda_n)}} K_{\lambda_n}$

$\Rightarrow |c_n| \sim \frac{1}{|\lambda_n| \cdot |f'(\lambda_n)|} \sim \frac{1}{\sqrt{n}} \Rightarrow \sum_n |c_n|^2 = \infty$. Contradiction! $\blacksquare$

Note: 最早的关于 Riesz basis 问题：对 λ_n 动多少仍然为 Riesz basis

Kadec $\frac{1}{4}$-thm 基本上完全回答了这种问题 $\frac{1}{4}$ 也为 sharp 的.

Remark. By refining the argument, we can prove that the set

不单单不为 Riesz basis 且连 basis 都不是

$$\{e^{\pm i(n-1/4)t} : n = 1, 2, 3, \ldots\}$$

is not even a *basis* for $L^2[-\pi,\pi]$ (see Problem 1).

3.4: Some Intrinsic properties of sets of complex exponentials

observe $f \in PW$ (or $f \in L^2$), $f(\mu) = 0$, then $\frac{z-\lambda}{z-\mu} f(z) \in PW$, More generally it holds in L^p

Thm 6. suppose $f(z) = \int_{-\pi}^{\pi} \alpha(t) e^{izt} dt$, $\alpha \in L^p[-\pi,\pi]$, $1 \le p < \infty$, $f(\mu) = 0$. and $g(z) = \frac{z-\lambda}{z-\mu} f(z)$ 换零点

then $\exists \beta \in L^p[-\pi,\pi]$, s.t. $g(z) = \int_{-\pi}^{\pi} \beta(t) e^{izt} dt$, In fact

$$\beta(t) = \alpha(t) + i(\lambda-\mu) e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds.$$

proof: Step 1. $\frac{1}{z-\mu} f(z) = \frac{1}{z-\mu} \int_{-\pi}^{\pi} \alpha(t) e^{i\mu t} \cdot e^{i(z-\mu)t} dt$ → for integration by parts

$$= \frac{1}{z-\mu} \int_{-\pi}^{\pi} e^{i(z-\mu)t} \, d\left[\int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds\right]$$

$$= -i \int_{-\pi}^{\pi} \underbrace{\left(e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds\right)}_{\alpha_1(t)} e^{izt} dt$$

Step 2. $(z-\lambda) \int_{-\pi}^{\pi} \alpha_1(t) e^{izt} dt$

$$= (z-\lambda) \int_{-\pi}^{\pi} \boxed{\alpha_1(t) e^{i\lambda t}} \cdot e^{i(z-\lambda)t} dt$$

$$= \frac{1}{i} \int_{-\pi}^{\pi} (\alpha_1(t) e^{i\lambda t}) \, d e^{i(z-\lambda)t}$$

$$= -i \int_{-\pi}^{\pi} e^{izt} (\alpha_1'(t) + i\lambda \alpha_1(t)) dt$$

$$-i\mu e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds + e^{-i\mu t} \alpha(t) \cdot e^{i\mu t}$$

$$+ i\lambda e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds$$

$$= -i \int_{-\pi}^{\pi} e^{izt} \underbrace{\left[\alpha(t) + i(\lambda-\mu) e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds\right]}_{=\beta(t)} dt \qquad ▥$$

Remark: It holds similarly on $C[a,b]$.

consequence of the above thm

Thm 7: The completeness of $\{e^{i\lambda_n t}\}$ in $L^p[-\pi,\pi]$, $1 < p < \infty$, or $C[a,b]$ is unaffected if one of λ_n is replaced by another.

Thm 8: The system of $\{e^{i\lambda_n t}\}$ in $C[a,b]$ (L^p) is complete, if and only if span$\{e^{i\lambda_n t}\}$ contains another exponent $e^{i\lambda t}$

Proof: "$\Rightarrow$" by completeness, trivial

"$\Leftarrow$" By translation, we may assume $\lambda = 0$ i.e. $1 \in$ span

We claim $f \in$ span $\Rightarrow \int_{-\pi}^{t} f \in$ span

If so $t, t^2, t^3, \cdots \in$ span, as desired, $\forall \varepsilon>0$, $\exists \sum_{n=1}^{N} c_n e^{i\lambda_n t}$, s.t. $\|f-\sum_{n=1}^{N} c_n e^{i\lambda_n t}\| < \varepsilon$

then $\|\int_{-\pi}^{t} f - \sum_{n=1}^{N} \frac{c_n}{i\lambda_n} e^{i\lambda_n t} - \sum_{n=1}^{N} \frac{c_n}{i\lambda_n} e^{-i\lambda_n t}\|$

$= \|\int_{-\pi}^{t} (f-\sum_{n=1}^{N} c_n e^{i\lambda_n t})\| \le \pi \cdot \|f-\sum_{n=1}^{N} c_n e^{i\lambda_n t}\| \le \pi \cdot \varepsilon.$ $\boxed{///}$

corollary of thm 8

Thm 9: Every incomplete set of $\{e^{i\lambda_n t}\}$ must be minimal.

$\text{span}_{n\neq n_0}\{e^{i\lambda_n t}\} \neq \text{span}_{n}\{e^{i\lambda_n t}\}$, $\forall n_0$

proof: Corollary of thm 8. $\boxed{///}$

前面thm的总结

Thm 10: $\{e^{i\lambda_n t}\}$ is either minimal or linked.

$\text{span}_{n\neq m}\{e^{i\lambda_n t}\} \neq \text{span}_{n}\{e^{i\lambda_n t}\}$ $\forall m$ $\quad$ $e^{i\lambda_m t} \in \text{span}_{n\neq m}\{e^{i\lambda_n t}\}$, $\forall m$

proof: Incomplete $\xRightarrow{\text{thm 9}}$ minimal

complete, if minimal $\checkmark$, or assume $\{e^{i\lambda_n t}\}$ is complete. but not minimal

$\exists n_0$ s.t. $\text{span}_{n\neq n_0}\{e^{i\lambda_n t}\} = \text{span}_{n}\{e^{i\lambda_n t}\}$

$\Rightarrow \{e^{i\lambda_n t}\}_{n\neq n_0}$ is complete.

by thm 7, $\{e^{i\lambda_n t}\}_{n\neq m}$, $\forall m \Rightarrow e^{i\lambda_m t} \in \text{span}_{n\neq m}\{e^{i\lambda_n t}\}$, $\forall m$ as desired. $\boxed{///}$

Lecture 10: Stability, Chapter 4: Interpolation and basis in Hilbert space

Lecture 10-2023 年 3 月 28 日星期二 天气阵雨，微冷，17°C-20°C

主要内容：结束了 Chapter 3 的内容（Stability），开始 Chapter 4，介绍了 Moment Sequences (Interpolation 问题[a])

其他信息：

[a]这里的 Interpolation 是指代类似 Lagrange Interpola 的插值问题。

there is only one section left in this chapter — stability

Section 3.5 : Stability

Recall thm 4: $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is complete if $|\lambda_n| \le n + \frac{1}{2p}$

perturbation of integers

general perturbation

In general, arbitrarily small perturbation does not preserve completeness

e.g. consider $\{e^{i\lambda_n t}\}_{n\neq 0}$ in $L^2[-\pi,\pi]$, $\lambda_n = \begin{cases} n-\frac{1}{4}, & n>0 \\ n+\frac{1}{4}, & n<0 \end{cases}$, we shall show that

this type of examples appears frequently!

① It is complete

② $\forall \varepsilon>0$, $\exists \tilde{\lambda}_n$, $|\tilde{\lambda}_n - \lambda_n| < \varepsilon$. but $\{e^{i\tilde{\lambda}_n t}\}$ is $\boxed{\text{not}}$ complete!

① To see the completeness, translate it by $\frac{1}{2}$ to obtain $\{\cdots, -2+\frac{3}{4}, -1+\frac{3}{4}, 1+\frac{1}{4}, 2+\frac{1}{4}, \cdots\}$
$\underset{=-1-\frac{1}{4}}{}$ $\underset{=0-\frac{1}{4}}{}$

i.e. $n\neq 0$, 少了一个要重新标号

then $|\lambda_n| \le n+\frac{1}{4}$ ✓ by thm 4, completeness.

② : $\forall\varepsilon>0$. let $\tilde{\lambda}_n = \begin{cases} n-\frac{1}{4}+\varepsilon, & n>0 \\ n+\frac{1}{4}-\varepsilon, & n<0 \end{cases}$, and $\tilde{\lambda}_0 = 0$ $(n=0)$, then $\{e^{i\tilde{\lambda}_n t}\} \cup \{e^{i\cdot 0\cdot t}\}$ is a Riesz basis

for $L^2[-\pi,\pi]$. by Kadec's $\frac{1}{4}$ thm. $\Rightarrow \{e^{i\tilde{\lambda}_n t}\}$ is incomplete 少了 $e^{i\cdot 0\cdot t}$ 一项

additional requirement of perturbation.

Thm 11 : If $\lambda_n, \mu_n \in \mathbb{R}$. $\boxed{\sum_n |\lambda_n - \mu_n| < \infty}$. then $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$. $1\le p<\infty$, then

$\{e^{i\mu_n t}\}$ is also complete!

∞ may also be okay!

relies on 接零点:

Proof : If not, $\exists\ \phi\neq 0 \in L^{p'}$ s.t. $f(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt}\,dt$ vanishes at all μ_n.

Recall that : if $f(z) = \int_{-\pi}^{\pi}\phi(t)e^{izt}\,dt$, $\phi\in L^{p'}$, $f(\mu)=0$, then thm 6

$$\frac{z-\lambda}{z-\mu} f(z) = \int_{-\pi}^{\pi} \beta(t) e^{izt}\,dt,\ \beta\in L^{p'}.$$

Denote $f_0 = f$, $f_n = \frac{z-\lambda_n}{z-\mu_n} f_{n-1}$, by thm 6

$$f_n(z) = \int_{-\pi}^{\pi} \phi_n(t) e^{izt}\,dt,\ \phi_n \in L^{p'},\ \text{and } \boxed{\phi_n(t) = \phi_{n-1} + i(\lambda_n-\mu_n)e^{-i\mu_n t}\int_{-\pi}^{t}\phi_{n-1}(s)e^{i\mu_n s}\,ds}$$

by Holder

$$\|\phi_n - \phi_{n-1}\| \le \underbrace{c\cdot|\lambda_n-\mu_n|}_{\text{by def } =\varepsilon_n}\cdot\|\phi_{n-1}\|$$

$$\Rightarrow (1-\varepsilon_n)\|\phi_{n-1}\| \le \|\phi_n\| \le (1+\varepsilon_n)\|\phi_{n-1}\|$$

RHS implies that $\|\phi_n\| \le \underbrace{\prod(1+\varepsilon_n)\|\phi_0\|}_{\text{finite, as } \sum\varepsilon_n<\infty} < \infty$

$$\Rightarrow \|\phi_{n+m} - \phi_n\| \le \sum_{k\ge n}\|\phi_{k+1}-\phi_k\|$$

$$\le \sum_{k\ge n}\varepsilon_k\|\phi_k\|$$

$$\le \underbrace{(\textstyle\sum_{k\ge n}\varepsilon_k)}_{\text{cauchy}}\cdot\underbrace{(\textstyle\prod(1+\varepsilon_n)\|\phi_0\|)}_{\text{finite}} \Rightarrow \{\phi_n\} \text{ cauchy}$$

then $\phi_n \to \tilde{\phi}$ in $L^{p'}$

indicates that $\{e^{i\mu_n t}\}$ is not complete!

It remains to show that $\tilde{\phi}\neq 0$, contradiction to completeness of $\{e^{i\lambda_n t}\}$

$$\|\phi_n\| \ge \underbrace{\prod_n(1-\varepsilon_n)}_{>0,\text{ as } \sum\varepsilon_n<\infty}\|\phi_0\| > 0$$

◻

End of chapter 3: the last section assume too much background information!

Chapter 4: Interpolation and bases in Hilbert space.

↳ e.g. the Lagrange Interpolation $\sum_i f(x_i) \left(\prod_{j\neq i} \frac{x-x_j}{x_i-x_j} \right)$

In Paley-Wiener space $PW[-\pi,\pi]$, give $\{c_n\} \in \ell^2$, $\exists!\ \phi \in L^2[-\pi,\pi]$, s.t.

$$\int_{-\pi}^{\pi} \phi(t) e^{-int} dt = c_n = f(-n), \text{ Fourier coefficients}$$

given $\{c_n\} \in \ell^2$, $n \in \mathbb{Z}$, then $\exists!\ f \in PW[-\pi,\pi]$ s.t. $f(n) = c_n$

In fact, $\forall f \in PW[-\pi,\pi]$, $f(z) = \sin(\pi z) \cdot \sum_{-\infty}^{\infty} (-1)^n \cdot \frac{f(n)}{\pi(z-n)}$

cardinal sequence

4.1. Moment sequences in $\mathcal{H}$

Def: Given $f_1, f_2, \cdots \in \mathcal{H}$, f fixed, we call (f, f_n) the n-th moment of f, and $\{(f, f_n)\}$ the moment sequence, and $\{((f, f_n))_n : f \in \mathcal{H}\}$ the moment space of $\{f_n\}$

Question: (1) Given $\{f_n\}$, $\{c_n\}$ is $\{c_n\} \in$ moment space?

i.e. $\exists ?\ f \in \mathcal{H}$ s.t. $(f, f_n) = c_n$?

(2) unique, if exists? ($\Leftrightarrow$ completeness of $\{f_n\}$)

(3) If not unique, how can the solutions be captured from $\{c_n\}$?

Proposition 1: If a solution exists, then $\exists!$ solution of minimal norm.

proof: unique in span $\{f_n\}$, $\boxed{[1]}$ P123

Example 1. (Finite interpolation) Given linearly independent $\{f_1, f_2, \cdots, f_n\}$, $c_1, \cdots, c_n$, then take

$$f = -\frac{1}{\det(f_i, f_j)} \det \begin{bmatrix} 0 & f_1 & \cdots & f_n \\ c_1 & & & \\ \vdots & & (f_i, f_j) & \\ c_n & & & \end{bmatrix}_{(n+1)\times(n+1)}$$

determined explicitly

notice $\det(f_i, f_j) \neq 0$, as $\langle (a_1, \cdots, a_n), (f_i, f_j) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rangle > \|\sum a_i f_i\|^2$

$$(f, f_k) = -\frac{1}{\det(f_i, f_j)} \det \begin{bmatrix} 0, & (f_1, f_k) & \cdots & , (f_n, f_k) \\ c_1 & & & \\ \vdots & & (f_i, f_j) & \\ c_n & & & \end{bmatrix}$$

$$= -\frac{1}{\det(f_i, f_j)} \det \begin{bmatrix} \boxed{-c_k} & & 0 & \\ c_1 & & & \\ \vdots & & \boxed{(f_i, f_j)} & \\ c_n & & & \end{bmatrix} = c_k$$

Example 2: If $\{f_n\}$ is a Riesz basis, then the moment space is ℓ^2, then $\forall (c_n) \in \ell^2$, one can take

$f = \sum c_n \underbrace{g_n}_{\text{bi-orthogonal sequence}} \in \mathcal{H}$

then $\langle f, f_n \rangle = c_n$

Example 3: Say $H = L^2[-\pi,\pi]$, and $\underbrace{\{e^{i\lambda_n t}\}}_{=\{f_n\}}$ is a Riesz basis, then what is $\underbrace{\{g_n\}}_{\text{biorthogonal sequence}}$?

We shall only consider the special case that $\lambda_{-n} = -\lambda_n \in \mathbb{R}$ (in particular $\lambda_0 = 0$)

product form of sin, $\frac{\sin \pi z}{\pi z}$

In this case $\boxed{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right) \in PW[-\pi,\pi]}$, vanishes at all λ_n but λ_0

assuming this for now, prove later

then $G_1(z) = z\cdot\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)$

$G(z)$ vanishes at all λ_n, $G_n = \frac{G_1(z)}{G_1'(\lambda_n)(z-\lambda_n)} \in PW$, vanishes at all λ_j but λ_n.

By Paley-Wiener thm. $\exists\, g_n \in L^2[-\pi,\pi]$, s.t.

$$\int_{-\pi}^{\pi} g_n(t)\, e^{-i\lambda_m t}\,dt = G_n(\lambda_m) = \delta_{n,m}$$

$\Rightarrow \{g_n\}$ is the bi-orthogonal sequence of $\{e^{i\lambda_n t}\}$

In fact, it is more convenient to translate this discussion to PW.

Then $\{e^{i\lambda_n t}\}$ is a Riesz basis in $L^2[-\pi,\pi]$

$\Updownarrow$

$\underbrace{\left\{\frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}\right\}}_{\text{reproducing kernel}}$ is a Riesz basis in $PW[-\pi,\pi]$

Then $\{G_n\}$ is the biorthogonal sequence of $\left\{\frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}\right\}$, as $\left(G_n, \frac{\sin\pi(z-\lambda_m)}{\pi(z-\lambda_m)}\right) = G_n(\lambda_m) = \delta_{n,m}$

So Given $\{c_n\}\in \ell^2$, $\left(f, \frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}\right) = c_n$

$\Longrightarrow$ the solution to this interpolation problem is

$$f(z) = \sum c_n G_n(z) = G_1(z)\sum \frac{c_n}{G_1'(\lambda_n)(z-\lambda_n)}$$ (also unique, by being a basis)

Furthermore, since $\left(f, \frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}\right) = c_n \Rightarrow = G_1(z)\sum\frac{f(\lambda_n)}{G_1'(\lambda_n)(z-\lambda_n)}$ (*)

$\Updownarrow$

$f(\lambda_n) = c_n$

Also notice $\{f(\lambda_n)\}\in\ell^2$, $\forall f\in PW$, so (*) is valid for all $f\in PW$

$\Downarrow$

a Generalization of Cardinal Series.

$\boxed{\text{It remains to show that } \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)\in PW[-\pi,\pi]}$ Since $\{e^{i\lambda_n t}\}_{n\neq0}$ is not complete, $\exists!\, f\in PW$ s.t. (also exponential type)

$f(\lambda_n) = \begin{cases} 0, & n\neq 0 \\ 1, & n=0\end{cases}$ of deficiency 1.

Claim: f must be a even function

Notice $\tilde f(z) = f(-z)$, also satisfies $\tilde f(\lambda_n) = \begin{cases}0, & n\neq0\\ 1, & n=0\end{cases}$

By Hadamard Factorization

$$f(z) = e^{Az+B}\prod_{n\neq0}\left(1-\frac{z}{\lambda_n}\right)e^{\frac{z}{\lambda_n}}$$

收敛的

$$= \lim_{N\to\infty}\prod_{\substack{n=-N\\ n\neq0}}^{N}\left(1-\frac{z}{\lambda_n}\right)e^{\frac{z}{\lambda_n}} \underset{\because f \text{ even}}{=} \prod_{n\geq1}\left(1-\frac{z^2}{\lambda_n^2}\right)$$

As $f(0)=1$, f is even, we have $B=0$, $A=0$.

$\Rightarrow f(z) = \prod_{n\geq1}\left(1-\frac{z^2}{\lambda_n^2}\right)\in PW$ ∎

The following is a general criteria on the existence of a solution

Thm 2: $(f, f_n) = c_n$, admits a solution f with $\|f\|\leq M \Longleftrightarrow \left|\sum a_n\bar{c}_n\right|\leq M\cdot\left\|\sum a_n f_n\right\|$

$\forall$ finite sequence $\{a_n\}$

proof: "$\Rightarrow$": $|\sum a_n \overline{c_n}| = |\sum a_n \overline{(f, f_n)}|$

$$= |(f, \sum a_n f_n)| \le \|f\| \cdot \|\sum a_n f_n\|$$

$$\le M \cdot \|\sum a_n f_n\|$$

"$\Leftarrow$": consider $T: \sum a_n f_n \to \sum a_n \overline{c_n}$, is a bounded linear functional on $\operatorname{span}\{f_n\}$

正交补部分令为0

of $\|T\| \le M$, It can be further extended to a bounded linear functional on $\mathcal{H}$.

By Riesz representation thm. $\exists f \in H$, $\|f\| \le M$, s.t. $(\sum a_n f_n, f) = \sum a_n \overline{c_n}$

$$\Rightarrow (f, f_n) = c_n$$

$\boxed{///}$

Exercise 5.

Lecture 11: cont.Bessel Sequences, Riesz-Fischer Sequences, Moment Space and Equivalent Sequences, Frame

Lecture 11-2023 年 3 月 30 日星期四 天气阴，微冷，20℃-24℃

主要内容：Bessel Sequence 和 Riesz-Fischer Sequence 的相关内容，下节课会介绍一些 Stability 的相关结论（本书最后的内容）

Stability 和 $\{\lambda_n\}$ separateness 有一些关系

其他信息：

We will finish this book within $\boxed{\text{2 classes}}$

Some notation (about Riesz basis) are still quite common in recent research

despite difference in seeing

less relies on complex analysis

In higher-dim. complex analysis is not that useful. (零点不孤立)

4.2 Bessel sequences and Riesz-Fischer sequences $(\{f_n\} \subset H)$

Def: (Bessel) $\sum |\langle f, f_n\rangle|^2 < \infty$, $\forall f \in H$

(R-F) $\forall (c_n) \in \ell^2$, $\exists f \in H$, s.t. $\langle f, f_n\rangle = c_n$

Equivalently: moment space of Bessel $\subset \ell^2 \subset$ moment space of R-F

Remark: "=" Riesz sequence = Bessel + R-F

Riesz sequence + completeness = Riesz basis.

Proposition 2: Bessel $\Leftrightarrow \underline{\sum |\langle f, f_n\rangle|^2 \le M \|f\|^2}$ 类似上界

by Banach-Steinhaus thm.

Riesz-Fischer $\Leftrightarrow \exists m > 0$, s.t. $\forall (c_n) \in \ell^2$, $\exists f \in H$, s.t. $\underline{\|f\|^2 \le \frac{1}{m} \sum |c_n|^2}$ 类似下界

uniformly bounded solution.

actually an exercise in the book.

Proof: consider $\ell^2 \xrightarrow{T} H / \operatorname{span}\{f_n\}^\perp$ (mod): well-defined, linear

we shall show that T is bounded.

Say $\alpha_k \to \alpha \in \ell^2$, $\alpha_k = ((c_{nk})_n$, $\alpha = (c_n)_n$.

$T\alpha_k \to \beta \in H / \operatorname{span}\{f_n\}^\perp$

$\langle \beta, f_n\rangle = \lim_{k\to\infty} \langle T\alpha_k, f_n\rangle = \lim_{k\to\infty} (c_{nk}) = c_n$

by the closed graph thm. T is bounded. $\square$

by def $T\alpha = \beta$

Thm 3: (1) $\{f_n\}$ is Bessel with bound M

$\Leftrightarrow \|\sum c_n f_n\| \le M \cdot \sum |c_n|^2$, $\forall$ finite sequence $\{c_n\}$

(2) R-F with bound $m \Leftrightarrow$

$m \sum |c_n|^2 \le \|\sum c_n f_n\|^2$, $\forall$ finite sequence $\{c_n\}$.

proof: If and only if

(1) Bessel: $T: f \mapsto (\langle f, f_n\rangle)$ is bounded by M.

$\langle T^*(b_n)_n, f\rangle = \sum b_n \overline{\langle f, f_n\rangle} = \langle \sum b_n f_n, f\rangle$

then it follows from $\|T\| = \|T^*\|$

(2) "$\Rightarrow$" let f be a solution of $(f, f_n) = c_n$.

$\|f\|^2 \le \frac{1}{m}\sum |c_n|^2$, then $m\cdot\sum|c_n|^2 = m\cdot\sum c_n\cdot\overline{(f, f_n)}$

$= m\cdot(\sum c_n f_n, f)$

$\le m\|\sum c_n f_n\|\cdot\|f\|$

$\overset{\text{prop2}}{\le} \sqrt{m}\,(\sum|c_n|^2)^{\frac{1}{2}}\cdot\|\sum c_n f_n\|$

then done.

last lecture

"$\Leftarrow$" Recall thm 2. $(f, f_n) = c_n$ has solution of norm $\le M$ if $|\sum a_n\overline{c_n}| \le M\cdot\|\sum a_n f_n\|$

$\forall$ finite sequence.

To check this condition, by Cauchy-Schwarz

$|\sum a_n\overline{c_n}|^2 \le \sum|a_n|^2\cdot\sum|c_n|^2$

$\le \frac{1}{m}\sum|c_n|^2\cdot\|\sum a_n f_n\|^2$

$< \infty$, as $(c_n)\in \ell^2$

$\overset{\text{by thm 2}}{\Rightarrow}$ $\exists$ a solution f s.t. $\|f\|^2 \le \frac{1}{m}\cdot\sum|c_n|^2$

$\boxed{//}$

In operator language,

Remark: Bessel of bound $M \Leftrightarrow T: e_n \to f_n$. $\|T\| \le \sqrt{M}$

$(\|\sum c_n f_n\|^2 \le M\cdot\underbrace{\sum|c_n|^2}_{=\|\sum c_n e_n\|^2})$

R-F of bound $m \Leftrightarrow S: f_n \to e_n$, $\|S\| \le \sqrt{\frac{1}{m}}$

In the language of Gram-matrix

$(f_i, f_j)_{ij} \overset{\text{def}}{=} A$, then

Bessel $\Leftrightarrow$ $\|A\| \le M$ on ℓ^2

R-F $\Leftrightarrow$ every $n\times n$ sub-matrix A_n of A satisfies $m\|c\|^2 \le \|A_n c\|$

$\forall\, c = (c_1, \cdots, c_n)$

Example: e.g. $\{1, t, t^2, \cdots\}$ is Bessel in $L^2[0,1]$, whose gram matrix $(\frac{1}{i+j+1})_{ij}$ that has norm π on ℓ^2, but not Riesz-Fischer, $\|f_n\| \geq c > 0$, while $\|t^n\| \to 0$.

Thm 4: If $\lambda_n \in \mathbb{R}$, separated $(|\lambda_n - \lambda_m| > \delta > 0, \forall n \ne m)$, then $\{e^{i\lambda_n t}\}$ is Bessel sequence in $L^2[-A, A]$, $\forall\, 0 < A < \infty$.

proof: $f \in PW$, then $f(z) = \int_{-A}^{A}\phi(t)e^{izt}dt$

$\sum|f(\lambda_n)|^2 \le c\int_{-\infty}^{+\infty}|f(x)|^2$

$\sum|(\phi, e^{i\lambda_n t})|^2 \qquad c\cdot\|\phi\|_{L^2}^2$

then we know it's a Bessel sequence. $\boxed{/}$

Section 4: The moment space and equivalent sequences.

Def: $\{f_n\}$ and $\{g_n\}$ are called equivalent if $\exists$ T bounded invertible $Tf_n = g_n$

~~only 1 thm in this section~~

Thm 7. Two complete sequences are equivalent if and only if they have the same moment space.

Cor: Completeness + Riesz sequence = Riesz basis (ℓ^2 为 moment space $\{e_n\}$)

(Riesz sequence: moment space ℓ^2)

Proof of thm 7: "$\Rightarrow$" $Tf_n = g_n$, Given $\langle f, f_n\rangle$. we need to find g s.t. $\langle g, g_n\rangle = \langle f, f_n\rangle$, $\forall n \in \mathbb{N}$ (Since T is invertible, only need to prove)

$$\langle f, f_n\rangle = \langle f, T^{-1}g_n\rangle = \langle \underbrace{(T^{-1})^* f}_{g}, g_n\rangle$$

then moment space of $\{f_n\}$ $\subseteq$ moment space of $\{g_n\}$

the other direction is similar.

"$\Leftarrow$" $\langle f, f_n\rangle = \langle g, g_n\rangle$ defines a bijection $f \leftrightarrow g$, and linear

We still need to show that it's bounded!

define $Tf = g$. use the $\boxed{\text{closed-graph thm}}$

We shall show that T is bounded, say $f_k \to f$, $Tf_k \to g$, then

$$\langle g, g_n\rangle = \lim_{k\to\infty} \langle Tf_k, g_n\rangle = \lim_{k\to\infty}\langle f_k, f_n\rangle = \langle f, f_n\rangle$$

$\Rightarrow$ $Tf = g$ as desired.

The other direction is similar $\Rightarrow$ T is invertible. Finally

$$\langle f, f_n\rangle = \langle Tf, g_n\rangle = \langle f, T^* g_n\rangle \Rightarrow T^* g_n = f_n$$ ///

Now, we will discuss stability of Riesz basis

still a popular topic in recent study.

Section 6: Interpolation in PW · stability

Def: $\{\lambda_1, \lambda_2, \cdots\} \subseteq \mathbb{C}$ is called an interpolating sequence, if

$$\{(f(\lambda_n))_n : f \in PW\} = \ell^2$$

$$\Updownarrow \quad \{(\int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t} dt)_n : \phi \in L^2[-\pi,\pi]\}$$

the moment space of $\{e^{-i\lambda_n t}\}$ is ℓ^2 $\Leftrightarrow$ $\{e^{-i\lambda_n t}\}$ is a Riesz sequence for $L^2[-\pi,\pi]$

If in addition, the solution for $f(\lambda_n) = c_n$ is unique. we call $\{\lambda_n\}$ complete interpolating sequence

$\Leftrightarrow$ $\{e^{-i\lambda_n t}\}$ is a Riesz basis for $L^2[-\pi,\pi]$.

Proposition: If $\{\lambda_1, \lambda_2, \cdots\} \subset \mathbb{C}$ is an interpolating sequence, then it must lie in a horizontal strip (对Im有要求), and be separated.

Proof: We first show it lies in a horizontal strip. Since it is Bessel, $\|\sum c_n f_n\|^2 \le M \cdot \sum |c_n|^2$

$$\Rightarrow \|f_n\|^2 \le M \text{, uniformly in } n$$

$$\int_{-\pi}^{\pi} e^{2\cdot \text{Im}z t} dt \sim \frac{e^{2\pi \cdot |\text{Im}z|}}{|\text{Im}z|}$$ bounded only if $|\text{Im}z|$ is bounded.

Say $|\text{Im}z| \le H$.

Then we prove it is separated. Since $\{e^{i\lambda_n t}\}$ is R-F, by prop-2, $\forall (c_n) \in \ell^2, \exists f$ s.t. $(f, e^{i\lambda_n t}) = c_n$, $\|f\|^2 \le \frac{1}{m} \sum |c_n|^2$

$\Rightarrow \forall m, \exists f_k$ s.t. $(f_k, e^{i\lambda_n t}) = \delta_{n,k}$, $\|f_k\| \le \frac{1}{m}$

Denote $F_k(z) = \int_{-\pi}^{\pi} f_k(t) e^{-izt} dt$, then $F_k(\lambda_n) = \delta_{n,k}$.

$$1 = |F_k(\lambda_n) - F_k(\lambda_k)| = \left| \int_{\lambda_k}^{\lambda_n} F_k'(z) dt \right|$$

$$\le |\lambda_n - \lambda_k| \cdot \sup_{|\text{Im}z| \le H} |F_k'|$$

Notice that $\sup |F_k'(z)| = \int_{-\pi}^{\pi} it f_k(t) e^{izt} dt$

$$\le \pi e^{H\pi} \|f_k\|_{L^2} \le \pi \cdot e^{H\pi} \cdot \frac{1}{m}$$

$\Rightarrow |\lambda_n - \lambda_k| \ge (\pi \cdot e^{H\pi} \cdot \frac{1}{m})^{-1} > 0 \Rightarrow$ separated! $\boxed{///}$

Our goal: If $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2[-\pi, \pi]$, then $\exists L > 0$, s.t. $\{e^{i\mu_n t}\}$ is also a Riesz basis if $|\mu_n - \lambda_n| < L$ recall it fails for completeness!

later, we first see section 7

Section 7: The theory of frame

Def: $\{f_n\} \subset H$, is called a frame, if $\exists A, B > 0$, s.t.

$$A\|f\|^2 \le \sum |(f, f_n)|^2 \le B\|f\|^2$$

(Riesz sequence: $A \sum |c_n|^2 \le \|\sum c_n f_n\|^2 \le B \sum |c_n|^2$)

Remark ① It's Bessel (by RHS) $\overset{\text{thm 3}}{\Longleftrightarrow} \|\sum c_n f_n\|^2 \le B \cdot \sum |c_n|^2$

② It must be complete. (by LHS)

③ union of frame is also a frame not a good property, frame 不一定为 separated, 没有 stability!

Example: ①: Every orthonormal basis is a frame

Recall that $L^2[-A, A] \xrightarrow{\text{extend}} L^2[-\pi, \pi]$
再找 Fourier 系数再限制回来
但 extension 不唯一
⇓
不一定为 basis

②: $\{e^{int}\}$ is a frame for $L^2[-A, A]$, $\forall A \le \pi$ A<π时 不一定为 basis.

More generally, frame for H is a frame for every subspace H', may not be a basis!

③: In PW, it means $A\int_{-\infty}^{\infty}|f(x)|^2dx \le \sum_n |f(\lambda_n)|^2 \le B\int_{-\infty}^{\infty}|f(x)|^2dx$

Now, give a frame $\{f_n\}$, consider $Tf = \sum \langle f, f_n\rangle f_n$

It's bounded as $\{f_n\}$ is Bessel $\Rightarrow \|\sum\langle f,f_n\rangle f_n\|^2 \le \sum|\langle f,f_n\rangle|^2 \le B\cdot\|f\|^2$

We shall show that T is invertible.

Notice $\langle Tf, f\rangle = \langle\sum\langle f,f_n\rangle f_n, f\rangle = \sum|\langle f,f_n\rangle|^2 \geq A\|f\|^2$

$\|Tf\|\|f\| \geq$ cauchy $\Rightarrow \|Tf\| \geq A\|f\|$

Also notice that T is self-adjoint

$\langle Tf, g\rangle = \sum\langle f,f_n\rangle\overline{\langle g,g_n\rangle} = \langle f, Tg\rangle$

If T is not onto, $\exists g \in \text{range}(T)^\perp \setminus \{0\}$

$\Rightarrow 0 = \langle T(Tg), g\rangle = \|Tg\|^2 \geq A^2\|g\|^2 > 0$, contradiction.

Hence T is onto, then by $\boxed{\text{open mapping theorem}}$

T is invertible and therefore $f = \sum\langle T^{-1}f, f_n\rangle f_n$

Lemma 5: Given a frame, $f = \sum a_n f_n$ is unique if we require $a_n = \langle g, f_n\rangle$, for some $g\in H$

Moreover, if $f = \sum b_n f_n$ for some other (b_n), then

$$\sum|b_n|^2 = \sum|a_n|^2 + \sum|b_n - a_n|^2 \quad (\geq \sum|a_n|^2)$$

Remark: Coefficents given by $a_n = \langle g, f_n\rangle$ is "minimal"

proof: Existence of $g\in H$ ✓

uniqueness: say $f = \sum\langle h, f_n\rangle f_n = \sum\langle T^{-1}(Th), f_n\rangle f_n$

$= Th$

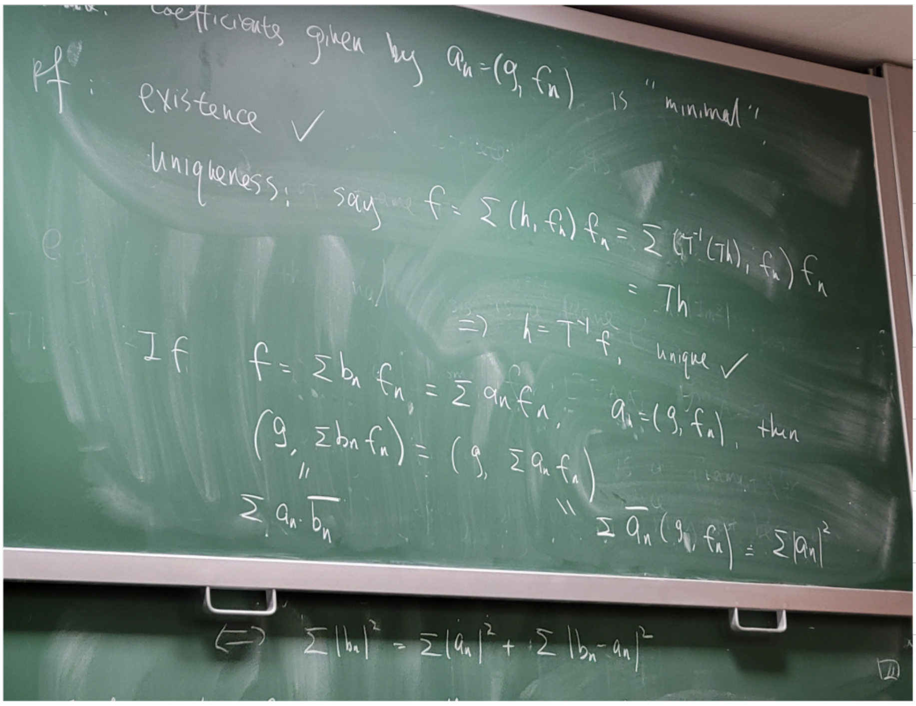
$\Rightarrow h = T^{-1}f$, unique

If $f = \sum b_n f_n = \sum a_n f_n$, $a_n = \langle g, f_n\rangle$, then

$$\langle g, \sum b_n f_n\rangle = \langle g, \sum a_n f_n\rangle$$

$$\sum a_n\overline{b_n} \qquad \sum\overline{a_n}\langle g, f_n\rangle = \sum|a_n|^2$$

$\Leftrightarrow \sum|b_n|^2 = \sum|a_n|^2 + \sum|b_n - a_n|^2$ ▯



Def: A frame is called exact if it fails to be a frame when any term is removed

We shall prove Riesz basis = exact frame

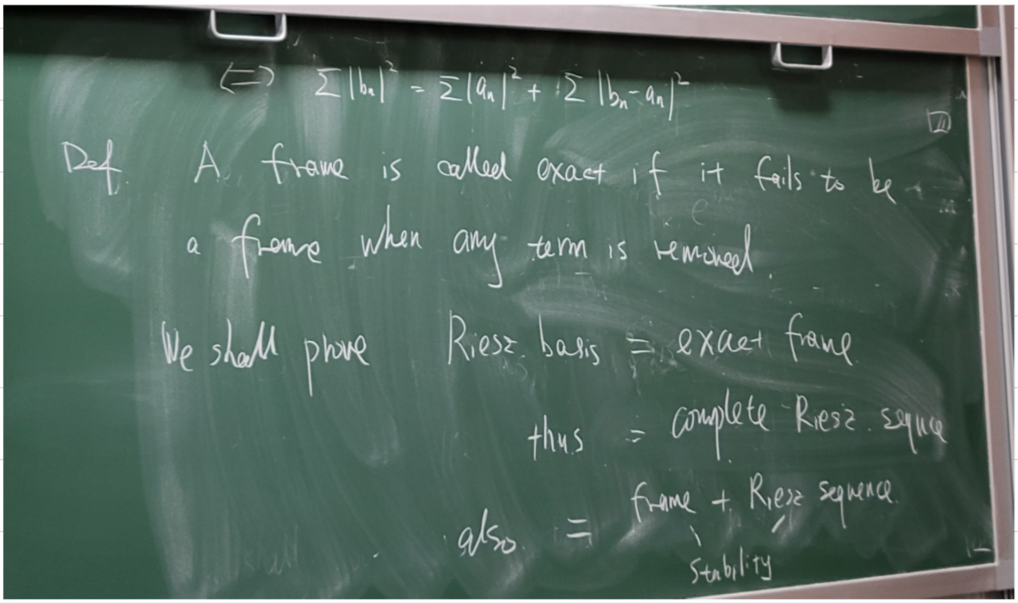
thus = complete Riesz sequence.

also = frame + Riesz sequence.

$\underbrace{\text{frame + Riesz sequence}}_{\text{Stability}}$

小测 (下下周)

↓ 简单 thm 应用 (all required thms, defs will be listed!)



Lecture 12: cont. Exact frame and Riesz basis, Stability of Non-harmonic Series

Lecture 12-2023 年 4 月 6 日星期四 小雨 ☂，闷热潮湿，18°C-27°C

主要内容：首先回顾了上节课的一些定义和基本结论，随后介绍了 Stability of frame，最后介绍了 series 的 pointwise-convergence 相关的一个定理作为本书的最后一个定理

在证明 Riesz-Fischer 的时候，把问题转换为 moment space 的问题

其他信息：下周二小测，大概三道题[a]

[a]参考本节课 sdocx 文件内录音

Recap: Bessel sequence : $\sum |(f, f_n)|^2 < \infty$, $\forall f \in H$, $\{f_n\} \subset H$

$\Leftrightarrow \sum |(f, f_n)|^2 \le M \|f\|^2$ (Banach-Steinhaus)

$\Leftrightarrow \|\sum c_n f_n\|^2 \le M \cdot \sum |c_n|^2$, $\forall$ finite sequence $\{c_n\}$

Riesz - Fischer : $\forall (c_n) \in \ell^2$, $\exists$ a solution $f \in H$ to the equations $(f, f_n) = c_n$

$\Leftrightarrow \exists m$, s.t. $\exists$ a solution $\|f\|^2 \le \frac{1}{m} \sum |c_n|^2$, $\forall (c_n) \in \ell^2$
uniformly.

f is unique with min-norm.

$\Leftrightarrow m \sum |c_n|^2 \le \|\sum c_n f_n\|^2$, $\forall$ finite sequence (c_n)

Def: Riesz sequence = Bessel + Riesz-Fischer, i.e.

$$m \sum |c_n|^2 \le \|\sum c_n f_n\|^2 \le M \cdot \sum |c_n|^2$$

$\Leftrightarrow$ moment space $\{((f, f_n)), f \in H\} = \ell^2$

$\{e^{i\lambda_n t}\}$ is a Riesz sequence for $L^2[-\pi, \pi]$

def $\{\lambda_n\}$ is an interpolating sequence for PW

+ completeness = Riesz basis.

Frame: $A\|f\|^2 \le \underbrace{\sum |(f, f_n)|^2 \le B\|f\|^2}_{\text{Bessel.}}$, complete, but not necessarily a basis.
$\{f_n\}$

Given a frame, $f = \sum (T^{-1}f, f_n) f_n$, T invertible.

$\exists !$ representation s.t. $f = \sum (g, f_n) f_n$,

$\forall$ representation $f = \sum a_n f_n$

$$\sum |(T^{-1}f, f_n)|^2 \le \sum |a_n|^2$$

Def (Exact frame) : we shall show that exact frame = Riesz basis $\leftarrow$ last lecture

Lemma 6 : the removal of a vector from a frame leaves either a frame or an incomplete set.

Proof: say we remove f_m. since $\{f_n\}$ is a frame, $\exists !$ $f_m = \sum_n (g_m, f_n) f_n$ see above recap

case ①: $(g_m, f_m) = 1$. by the "minimality" of $\sum (g_m, f_n) f_n$,

$\sum |(g_m, f_n)|^2 \le 1 \longrightarrow f_m = \sum \delta_{n,m} f_n$

$= 1 + \sum_{m \ne n} |(g_m, f_n)|^2 \Rightarrow (g_m, f_n) = \delta_{n,m} \Rightarrow \{f_n\}_{n \ne m}$ is incomplete.

其他f_n的线性组合

case ②: $(g_m, f_m) \ne 1 \Rightarrow f_m = \sum_{n \ne m} b_n f_n$, $0 < \sum |b_n|^2 < \infty$

then $\forall f \in H$. $|(f, f_m)|^2 = |\sum_{n \ne m} b_n (f, f_n)|^2 \le \sum |b_n|^2 \cdot \sum_{n \ne m} |(f, f_n)|^2$.

Now we show that $\{f_n\}_{n\neq m}$ is a frame. Upper bound is trivial.

$$\sum_{n\neq m}|\langle f,f_n\rangle|^2 \le \sum_n |\langle f,f_n\rangle|^2 \le B\|f\|^2$$

For lower bound.

by above. in ②

$$A\|f\|^2 \le \sum_n |\langle f,f_n\rangle|^2 = \sum_{n\neq m} \sim + |\langle f,f_m\rangle|^2 \le \boxed{\left[1+\sum|b_n|^2\mu\right]}\cdot \sum_{n\neq m}|\langle f,f_n\rangle|^2$$

$\Rightarrow \{f_n\}_{n\neq m}$ is a frame.

Remark: Give an exact frame, then $\{f_n\}$, then $\{f_n\}_{n\neq m}$ is not a frame case ①不成立

$\Rightarrow \langle g_m,f_n\rangle=\delta_{n,m}$ by the proof above.

$\Rightarrow$ Every exact frame admits a bi-orthogonal sequence. useful in inner-product representation

Thm 12: Exact frame = Riesz basis

"$\Leftarrow$": $\exists$ invertible T, s.t. $Te_n=f_n$, so

$$\sum|\langle f,f_n\rangle|^2=\sum|\langle T^*f,e_n\rangle|^2=\|T^*f\|^2\approx\|f\|^2$$

$\Rightarrow \{f_n\}$ is a frame.

As $\{f_n\}_{n\neq m}$ must be incomplete, thus not a frame, Overall Riesz basis is an exact frame.

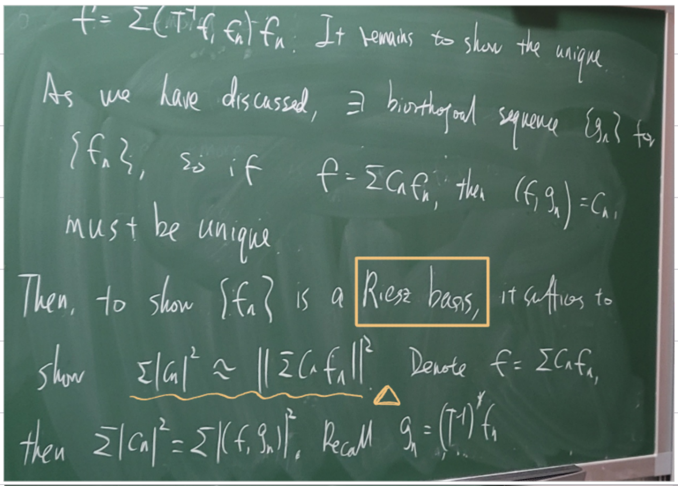
"$\Rightarrow$" We first show that it's a basis, since it's a frame, $\boxed{f=\sum\langle T^{-1}f,f_n\rangle f_n}$ check recap.

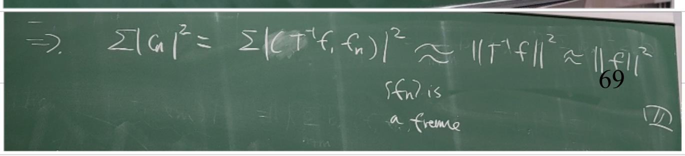
It remains to show that it's unique.

As we have discussed, $\exists$ biorthogonal sequence $\{g_n\}$ for $\{f_n\}$

So if $f=\sum c_nf_n$, then $\langle f,g_n\rangle=c_n$, must be unique.

Then to show that $\{f_n\}$ a Riesz basis, it sufficies to show





Section 8: $\boxed{\text{Stability}}$ of Non-harmonic Series.

Recall Riesz basis $=\underbrace{\text{Riesz sequence} + \text{frame}}_{\text{both Bessel}}$.

we first show a result of Bessel sequence.

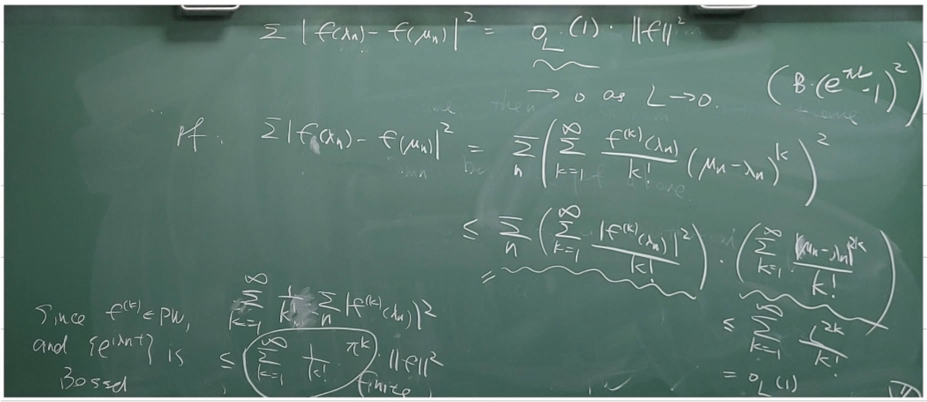
a simpler version, compared to the lemma in textbook.

Lemma 3: If $f\in PW$, $\sum|f(\lambda_n)|^2\le B\cdot\|f\|^2$, then $\forall\ \{\mu_1,\cdots\}$, $\sup_n|\lambda_n-\mu_n|\le L<\infty$, then

$$\sum|f(\lambda_n)-f(\mu_n)|^2=\underbrace{o_L(1)}_{\to 0,\ \text{as } L\to 0}\cdot\|f\|^2$$

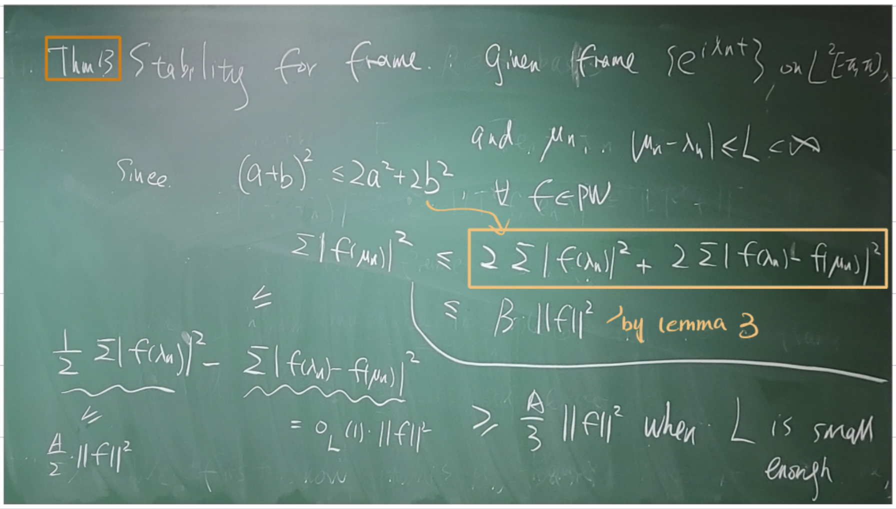
proof: $\sum_n|f(\lambda_n)-f(\mu_n)|^2=\sum_n\left(\sum_{k=1}^{\infty}\frac{f^{(k)}(\lambda_n)}{k!}(\mu_n-\lambda_n)^k\right)^2$

$$\le\sum_n\left(\sum_{k=1}^{\infty}\frac{|f^{(k)}(\lambda_n)|^2}{k!}\right)\cdot\underbrace{\left(\sum_{k=1}^{\infty}\frac{|\mu_n-\lambda_n|^{2k}}{k!}\right)}_{\le\sum_{k=1}^{\infty}\frac{L^{2k}}{k!}=O_L(1)}$$



Now, the $\boxed{\text{stability}}$ of frame: (Thm B) Given frame $\{e^{i\lambda_n t}\}$ on $L^2[-\pi,\pi]$, and μ_n, $|\mu_n-\lambda_n|\le L<\infty$

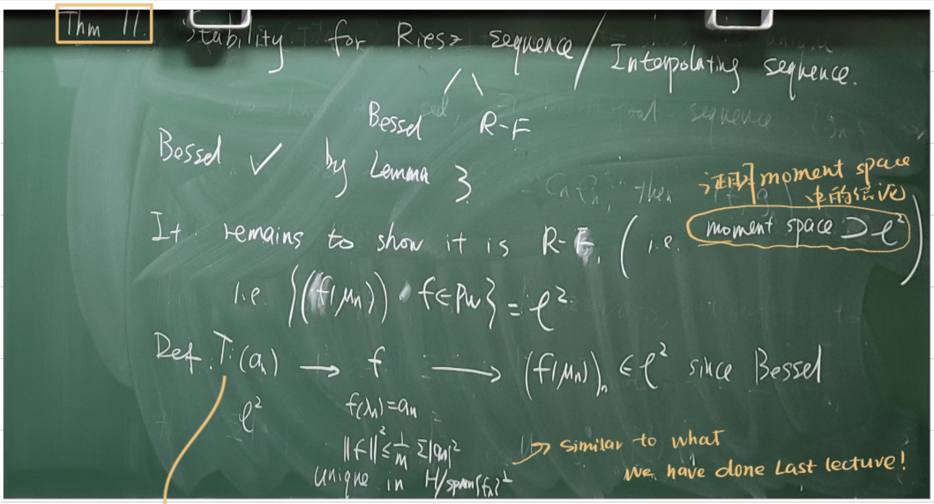
Since $(a+b)^2\le 2a^2+2b^2$, $\forall f\in PW$



Now for stability of Riesz sequence / Interpolating sequence. (Thm 11)

$\overbrace{\text{Bessel}\quad\text{R-F}}$

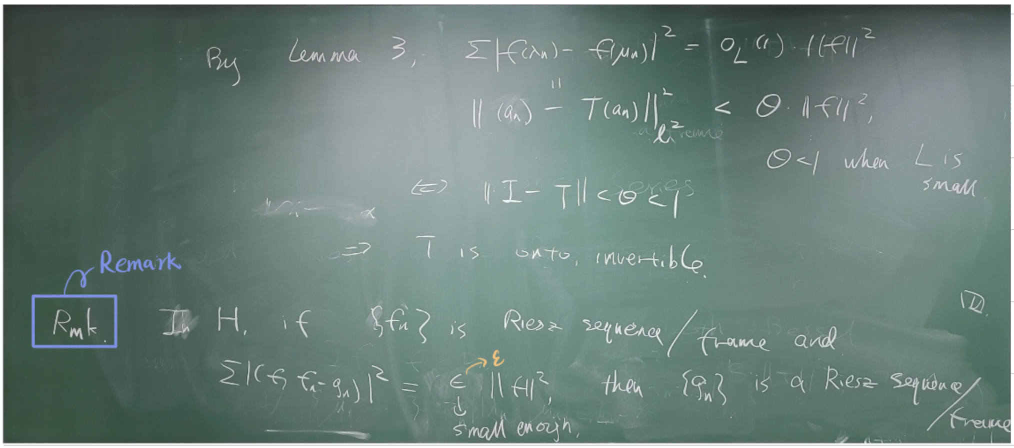
Done by lemma 3.



By lemma 3. $\sum |f(\lambda_n) - f(\mu_n)|^2 = O_L(1)\cdot \|f\|^2$

$\|(a_n) - T(a_n)\|_{\ell^2}^2 < \theta \cdot \|f\|^2$. $\theta<1$. when L is small.

$\Leftrightarrow \|I-T\| < \theta < 1 \Rightarrow T$ is onto, invertible! (see previous lecture!)



by stability, we may assume $\{\lambda_n\}$ as rational numbers!

Now. we show the last theorem in this book.

4.9: Pointwise-convergence.

Def: $\sum a_n$, $\sum b_n$ are said to be $\boxed{\text{equi-convergent}}$ if $\sum_{n=1}^{N}(a_n - b_n) \to 0$

proof is a little complicated.

Thm 15: If $\{e^{i\lambda_n t}\}$ is a $\boxed{\text{Riesz basis}}$ for $L^2[-\pi,\pi]$, and $\sup_n |\lambda_n - n| \overset{\text{def}}{=} L < \infty$, then $\forall f \in L^2[-\pi,\pi]$.

the ordinary Fourier Series and non-harmonic Fourier series are uniformly equi-convergent on every cpt subset of $(-\pi,\pi)$.

proof: $f = \sum a_n e^{int} = \sum c_n e^{i\lambda_n t}$, we need to estimate the partial sum

$$\sum_{-N}^{N} (a_n e^{int} - c_n e^{i\lambda_n t})$$

To do this, we shall find a good representation.

write $e^{i\lambda_n t} = e^{int} \cdot e^{i(\lambda_n - n)t}$

Pf: $f = \sum a_n e^{int} = \sum c_n e^{i\lambda_n t}$. We need to estimate the partial sum $\sum_{-N}^{N}(a_n e^{int} - c_n e^{i\lambda_n t})$. To do this we ~~shall~~ find a good representation.

Write $e^{i\lambda_n t} = e^{int}\cdot e^{i(\lambda_n - n)t}$

$$= e^{int}\boxed{\sum_{k=0}^{\infty}\frac{(i(\lambda_n-n)t)^k}{k!}}$$

$$:= e^{int}\sum_{k=0}^{\infty} b_{nk}\, t^k,$$ then

Then $f_N := \sum_{-N}^{N} c_n e^{i\lambda_n t}$

$$= \sum_{-N}^{N} c_n e^{int}\cdot\sum_{k=0}^{\infty} b_{nk} t^k$$

$$= \sum_{k=0}^{\infty}\underbrace{\left(\sum_{n=-N}^{N} c_n b_{nk} e^{int}\right)}_{:=\psi_{Nk}} t^k \quad \text{def} = \psi_{N,k}$$

We shall show $f = \sum_{k=0}^{\infty}\left(\sum_{n=-\infty}^{\infty} c_n b_{nk} e^{int}\right) t^k$ → Goal. To see this,

① We first show that (内层) $\lim_{N\to\infty}\psi_{Nk} = \psi_k \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} c_n b_{nk} e^{int}$, in L^2

this is because.

This is because $(c_n) \in \ell^2$, and $|b_{nk}| \le \frac{L^k}{k!} < \infty$ uniformly in k.

$\Rightarrow (c_n b_{nk})_n \in \ell^2$, so $\psi_{Nk} \to \psi_k$ in L^2

② Second (外层), we show $\sum \psi_k t^k$ → f is Cauchy (cauchy) in L^2

Notice $\|\psi_k\|_{L^2}^2 = \sum_n |c_n|^2 \cdot |b_{nk}|^2 \le \frac{L^{2k}}{(k!)^2}\sum |c_n|^2$

$$\Rightarrow \left\|\sum_{k>M}\psi_k t^k\right\| \le \sum_{k>M}|\pi|^k\cdot\|\psi_k\| \le \sum_{k>M}\frac{(\pi L)^k}{k!}\cdot\left(\sum_n |c_n|^2\right)^{\frac12}$$ cauchy

取端点 π

then

③ $f = \sum \psi_k t^k$. To see this, consider

$$\left\|\sum \psi_k t^k - f_N\right\| = \left\|\sum_{k=0}^{\infty}\psi_k t^k - \sum_{k=0}^{\infty}\sum_{k=-N}^{N} c_n b_{nk} e^{int} t^k\right\|$$

$$= \left\|\sum_{k=0}^{\infty}\left(\sum_{|k|>N} c_n b_{nk} e^{int}\right) t^k\right\|$$

取端点，与②类似

$$\le \sum_{k=0}^{\infty}\frac{(\pi L)^k}{k!}\cdot\sum_{n>|N|}|c_n| \to 0$$

Then, recall the $\boxed{\text{Dirichlet kernel}}$ →部分和的内积表达 c.f. stein-complex $D_N(t) = \frac{\sin(N+\frac12)t}{\sin\frac12 t}$. Say

$f = \sum a_n e^{int}$, then $\sum_{-N}^{N} a_n e^{int} = (f(x), D_N(x-t))$ →积分

求和 积分 → 交换求和与积分

Now our $f = \sum_{k=0}^{\infty} \psi_k t^k$, so

$$\boxed{\sum_{n=-N}^{N} a_n e^{int} = \sum_{k=0}^{\infty} \left(\psi_k(x), x^k D_N(x-t) \right)}$$ harmonic

On the other hand, ψ_{Nk} is the N-th partial sum of ψ_k,

so $\psi_{Nk}(t) = \left(\psi_k(x), D_N(x-t) \right)$ and

$$\boxed{\sum_{n=-N}^{N} c_n e^{i\lambda_n t} = f_N = \sum_{k=0}^{\infty} \psi_{Nk} t^k = \sum_{k=0}^{\infty} \left(\psi_k(x), t^k D_N(x-t) \right)}$$ non-harmonic

therefore

Therefore, $\sum_{n=-N}^{N} \left(a_n e^{int} - c_n e^{i\lambda_n t} \right) = \sum_{k=1}^{\infty} \left(\psi_k(x), \underbrace{(x^k - t^k) D_N(x-t)}_{\text{uniformly bounded in } N \text{ and } \boxed{|t| \le \pi - \delta,}\ \forall \delta > 0} \right)$

需要让$|t| \le \pi$，∵ $\sin\frac{1}{2}t \neq 0$ of D_N

再由 fixed M let $N \to \infty$

先$\varepsilon \to M$充分大

$$= \sum_{k \le M} + \sum_{k > M}$$

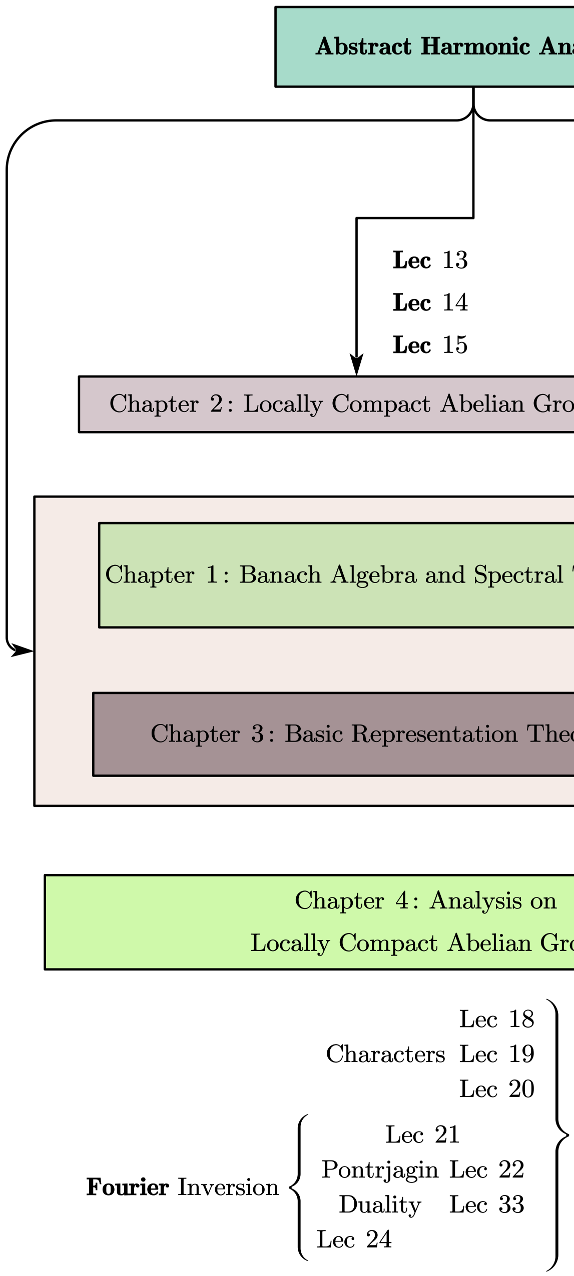
By approximating ψ_k by C_0^∞-function, $\left| \lim_{N \to \infty} \sum_{k \le M} \right| \le \epsilon$, thanks to the oscillation of $\sin(N + \frac{1}{2})t$ by integration by parts

For $\sum_{k > M}$, $\left| \sum_{k > M} \right| \le C_\delta \cdot \sum_{k > M} \|\psi_k\|_1 (2\pi)^k < \epsilon$ when M is large enough $\square$

下周2，3个题

Part 2: A Course in Abstract Harmonic Analysis

Part 2 主要包含个 Chapters：



Lecture 13: Chapter 2: Locally Compact Group, Haar measure, Midterm

Lecture 13-2023 年 4 月 11 日今天略潮湿 22°C-27°C

主要内容：今天开始进入 Part 2（抽象调和分析），首先介绍了 Locally Compact Topological Group，随后介绍了其上的一个测度（Haar 测度），分为 Left Haar measure 和 Right Haar measure 分别对应不同的不变性。

- If f is a function on the topological group G, and $y \in G$, we define the left and right translates of f through y by

$$L_y f = f(y^{-x}x), R_y f(x) = f(xy).$$

The reason for using y^{-1} in L_y and y in R_y is to make the maps $y \mapsto L_y, y \mapsto R_y$ group homomorphisms: $L_{yz} = L_y L_z, R_{yz} = R_y R_z$.

其他信息：今日 Midterm 考察了三个题目（主要问的是例子，以及一道关于 entire function of type 0 结合 Liouville 定理的问题），从 8 点 50 下课后开始考试到 9:50 分结束。

Ref: "A course in Abstract Harmonic Analysis"

Chapter 2: locally compact group

2.1: Topological group

def: G. group, topological space s.t. $G\times G \to G$, $(x,y)\mapsto x\cdot y$; $x\mapsto x^{-1}$, $G\to G$ } both continuous

Example: $(\mathbb{R}^n,+)$, $\mathbb{R}_\times = ((0,\infty),\times)$,

$GL_n(\mathbb{R})$, SL_n, $\mathbb{T}=\mathbb{R}/\mathbb{Z}\sim S^1$ } both are Lie groups

↳ we may skip most content about Lie group

Notation: Denote by 1, the identity

$\forall x,y\in G$, $A,B\subseteq G$, denote $x\cdot A \overset{\text{def}}{=} \{xa : a\in A\}$, similarly $B\cdot y$

$A\cdot B = \{a\cdot b : a\in A, b\in B\}$

$A^{-1}=\{a^{-1} : a\in A\}$

neighborhood, compact neighborhood: open with compact closure.

locally compact

Prop 2.1: (a) Translations and inversion are homeomorphism

Moreover U is open $\Rightarrow U\cdot A$ is open, $\forall A\subset G$ ✓ skip the proof

(b) $\forall$ neighborhood U of 1, $\exists$ a symmetric neighborhood V of 1 s.t. $V\cdot V\subset U$

proof of (b): $(x,y)\mapsto x\cdot y$ is continuous

$W\times W\in$ pre-image of U, then take $V=W\cap W^{-1}$ (Common Argument)

(c) If H is a subgroup of G, so is $\overline{H}$

proof of (c): $\forall \bar{x},\bar{y}\in\overline{H}$, consider $\bar{x}\cdot\bar{y}$

$\forall$ neighborhood U of $\bar{x}\cdot\bar{y}$, by continuity of $(x,y)\mapsto x\cdot y$, $\exists$ neighborhood W_1 of $\bar{x}$

and neighborhood W_2 of $\bar{y}$, s.t. $W_1\cdot W_2\subset U$

Similarly for $\bar{x}^{-1}$.

(d) Every open subgroup of G is closed. ✓ by considering coset.

(e) A,B compact $\Rightarrow AB$ compact ✓ compactness is preserved by continuity

Suppose H is a subgroup of G, then $q: G\to G/H$, with quotient topology

i.e. $V\subseteq G/H$ is open if and only if $q^{-1}(V)$ is open in G.

Notice that q is an open mapping, $q^{-1}(q(U)) = U\cdot H$ is open if U is open.

Proposition 2.2 (a) If H is closed, then G/H is Hausdorff.

(b) If G is locally cpt. so is G/H ← Easy. by quotient mapping, skip the proof.

(c) If H is normal, then G/H is a topological group.

Proof: (a)

$\forall \bar{x} \neq \bar{y} \in G/H$. consider xHy^{-1}, well-defined (closed) and $1 \notin xHy^{-1}$

$\Rightarrow (xHy^{-1})^c$ is a neighborhood of 1

$\Rightarrow \exists$ symmetric U of 1 s.t. $U \cdot U \cap xHy^{-1} = \emptyset$

$\Rightarrow \underbrace{UxH}_{\text{开集}} \cap \underbrace{UyH}_{\text{开集}} = \emptyset \Rightarrow$ Hausdorff.

(c): It's easy to see that G/H is a group., only need to show that multiplication & inverse are continuous.

Now we show that $(\bar{x}, \bar{y}) \to \bar{x}\bar{y}$ is continuous.

$\forall$ neighborhood U of $\bar{x}\bar{y} \to q^{-1}(U)$ is a neighborhood of $x \cdot y$

$\Rightarrow \exists$ a neighborhood W_1 of x, neighborhood W_2 of y such that

$W_1 \cdot W_2 \subset q^{-1}(U) \Rightarrow q(W_1 \cdot W_2) \subset U$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad || $

$q(W_1) \cdot q(W_2)$, as $q(x) \cdot q(y) = q(x \cdot y)$

For inversion, $\forall$ neighborhood U of $q(x)$, notice $q(q^{-1}(U)^{-1}) = U^{-1}$ ◻

$\uparrow$

As $(q^{-1}(U))^{-1} = (U \cdot H)^{-1} = H \cdot U^{-1} = U^{-1} \cdot H$ by normed.

Corollary 2.3: ① If G is T_1 (单点闭集), then G is Hausdorff

proof of ①: $G = G/\{1\}$. by abouve proposition.

②: If G is not T_1, then $\overline{\{1\}}$ is a closed normal subgroup, and then $G/\overline{\{1\}}$ is a Hausdorff topological group

Proof of ②: First $\overline{\{1\}}$ is the smallest closed subgroup of $G \Rightarrow \overline{\{1\}}$ is normal

then by Prop 2.2 $G/\overline{\{1\}}$ is a Hausdorff topological group. ◻

Prop 2.4: Every locally compact group G has a subgroup G_0. open, closed, σ-compact

Corollary If G is connected, then G is σ-compact

proof of cor: G connected $\Rightarrow G_0 = G$, then G is σ-compact

proof of the prop 2.4:

$\exists$ a symmetric compact neighborhood V of 1, let $G_0 = \bigcup_{n \geq 1} V^n$, subgroup. open, closed.

then $\overline{V^n} \subset G_0 \Rightarrow G_0 = \bigcup_{n \geq 1} \overline{V^n}$, σ-compact! $\boxed{//}$

Def: let f be a function on G. define $L_y f(x) = f(y^{-1}x)$

$$R_y f(x) = f(xy)$$

then $L_{xy} = L_x L_y$. $R_{xy} = R_x R_y$

We say f is left (right) uniformly continuous, if $\|L_y f - f\|_{sup} \to 0$ uniformly as $y \to 1$

$(\|R_y f - f\|_{sup} \to 0)$

Prop 2.6: If $f \in C_c(G)$, then f is left and right uniformly continuous.

similar to the one in mathematics analysis

2.2 Haar measure

Def: A left (right) Haar measure is a non-zero Radon measure on G s.t. $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for all Borel sets $E \subset G$. $\forall x \in G$

Radon: finite on cpt sets. inner (compact) and outer (open) regularity

Example: Lebesgue measure on $(\mathbb{R}^n, +)$

$\mu = \frac{dx}{|x|}$ on $(\mathbb{R}\setminus\{0\}, \times)$, where $\mu(E) = \int \chi_E(x) \cdot \frac{1}{|x|} dx$

$\frac{dxdy}{x^2+y^2}$ on $(\mathbb{C}\setminus\{0\}, \cdot)$

$\Bigg\}$ both left and right Haar measure.

Remark: ① If μ is left (right) invariant then $\hat{\mu}(E) \overset{def}{=} \mu(E^{-1})$ is right (left) invariant.

② μ is left haar $\Leftrightarrow$ $\int L_y f(x)\, d\mu(x) = \int f(x)\, d\mu(x)$, $\forall y$, $\forall f \in C_c^+(G)$

积分形式的定义

$$\int L_y f(x)\, d\mu(x) = \int f(y^{-1}x)\, d\mu(x) = \int f(x)\, d(L_y)_* \mu(x)$$

③ $\mu(U) > 0$, if U is an open set of non-empty interior.

proof: we may assume $1 \in U$, then $\mu(U) = 0 \Rightarrow$ every compact set has measure 0

$\Rightarrow$ Every set has measure 0 by inner regularity

$\Downarrow$

contradiction (non-zero measure)

Important! (存在唯一性)

Thm 2.10: Every locally compact group G posses a left Haar measure. Moreover if μ, ν are left Haar measure, then $\mu = c\nu$ for some constant c.

Remark: the same holds for "right".

More examples: (P45) $\prod_{i<j} dx_{ij}$ (Lebesgue measure) on $\{(x_{ij})_{n\times n} : x_{i,i}=1, x_{ij}=0, \forall i>j\}$ $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

on $GL_n(\mathbb{R})$, $|\det(T)|^{-n} \prod_{i,j=1}^{n} dx_{ij}$: $T(x_1,\cdots,x_n) = (Tx_1, Tx_2, \cdots, Tx_n)$ } left and right invariances.
$\ni T$

Now we give an example on which left and right Haar measures are different

$G_1 \overset{\text{def}}{=} \{g : gx = ax+b,\ \forall x\in\mathbb{R},\ \text{for some } a>0,\ b\in\mathbb{R}\} = \{(a,b)\in \mathbb{R}_+\times\mathbb{R}\}$

then its left Haar measure : $\frac{da\,db}{a^2} = \mu$

right Haar measure. $\frac{da\,db}{a} = \nu$,

- In G_1 $(a,b).(c,d)x$

$=(a,b)(cx+d)$

$= acx + ad + b \leadsto (a,b)\cdot(c,d) = (ac, ad+b)$

- $1 = (1,0)$, $(a,b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$,

$\boxed{\int L_{(c,d)} f(a,b)\, d\mu}$

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(\underbrace{(c,d)^{-1}(a,b)}_{=(\frac{1}{c},-\frac{d}{c})\cdot(a,b) = (\frac{a}{c},\frac{b}{c}-\frac{d}{c})})\frac{da\,db}{a^2}$

$= \int_{-\infty}^{\infty}\int_0^{\infty} \underbrace{f(\frac{a}{c}, \frac{b}{c} - \cancel{\frac{d}{c}})}_{\text{translation}} \frac{da\,db}{a^2}$

let $a' = \frac{a}{c}$, $b' = \frac{b}{c}$, then

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(a',b')\frac{da'db'}{(a')^2} = \int f\, d\mu \Rightarrow$ left-invariant $\Rightarrow \mu$ is left-invariant.

Now for ν $\int R_{(c,d)} f(a,b)\, d\nu(a,b)$

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(ac, \cancel{ad}+b)\frac{da\,db}{a}$ translation

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(a',b')\cdot\frac{da'db'}{a'} = \int f\, d\nu \Rightarrow \nu$ is right-invariant.

Remark: • Lie group: left-invariant volume form. 不变的测度

compact上 left, right都一样

• If G_α compact, with probability left Haar measure λ_α, then one can construct

a left Haar measure λ on $\prod_\alpha G_\alpha$ by extension of

$I(f) = \int\cdots\int f(x_{\alpha_1},\cdots,x_{\alpha_n})\, d\lambda_{\alpha_1}\cdots d\lambda_{\alpha_n}$

$f\in C(G)$, $f=1$ for all λ_α, but $\alpha_1,\cdots,\alpha_n$

In particular $\mathbb{Z}_2^{\mathbb{N}}$ equivalently $\mathbb{Z}_2^{\mathbb{N}} \to [0,1]$

$(a_1,a_2,\cdots)\longmapsto \sum_{j=1}^{\infty} a_j\frac{1}{2^j}$.

Lecture 14: p-adic field $\mathbb{Q}_p$

Lecture 14-2023 年 4 月 13 日今天干燥舒适 22°C-26°C

主要内容：首先引入了 Q 中的一个新的 metric $|\cdot|_p$，随后介绍了 modular function

- Modular function: 描述左不变的 Haar 测度和右不变的 Haar 测度到底差多少

$$\int f(x,y)\,d\lambda(x) = \Delta(y^{-1}) \int f(x)\,dx.$$

其他信息：今日开始尝试全程笔记录音[a]

[a] 效果还不错

Important example $\boxed{\mathbb{Q}_p}$: the field of p-adic numbers

$\forall r \in \mathbb{Q} \setminus \{0\}$, $r = p^m \cdot \frac{a}{b}$, $(a,b)=1$, $p \nmid ab$

$\downarrow$

then we define p-adic norm of r. $|r|_p = p^{-m}$. In addition, define $|0|_p = 0$

Notice $|r_1 + r_2|_p \le \max(|r_1|_p, |r_2|_p)$

$|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$

Now $d(r_1, r_2) \overset{def}{=} |r_1 - r_2|_p$ defines a metric on $\mathbb{Q}$

$\downarrow$

denote by $\mathbb{Q}_p$ its completion, called the field of p-adic members.

Prop 2.8 If $m \in \mathbb{Z}$, $c_j \in \{0, 1, \cdots, p-1\}$ for $j \ge m$, then every sequence $\sum_{j \ge m} c_j p^j$ (级数展开形式) is ① convergent in $\mathbb{Q}_p$.

moreover, every p-adic number is the sum of such a series

Proof: $|\sum_{j=n}^{N} c_j p^j|_p \le p^{-n} \to 0$, as $n \to \infty$ $(\forall N)$, thus the sequence is Cauchy

$\Downarrow$

① holds. convergent in $\mathbb{Q}_p$.

On the other hand, it sufficies to show

$\{\sum_{j=m}^{\infty} c_j p^j, m \in \mathbb{Z}, c_j = \{0, 1, \cdots, p-1\}\} \subseteq \mathbb{Q}_p$ is a field ~~field~~ containing, that is complete under $|\cdot|_p$

then by completion 的最小性

We first show that it is $\boxed{\text{complete}}$ under $|\cdot|_p$

$\forall$ Cauchy sequence $\sum_{j \ge m_n} c_{j,n} p^j$, then $\forall M > 0$, $\exists N$ s.t.

$|\sum_{j \ge m_{n_1}} c_{j,n_1} p^j - \sum_{j \ge m_{n_2}} c_{j,n_2} p^j|_p \le p^{-M}$, $\forall n_1, n_2 > N$

$\Downarrow$

$c_{j,n_1} = c_{j,n_2}$ for all $j \ge N$

$\Rightarrow c_j, j \ge m$, s.t. $\forall j$, $\exists N_1$ s.t. $c_j = c_{j,n}$, $n > N$

$\Rightarrow \sum c_j p^j$ is the limit. $\Rightarrow$ complete.

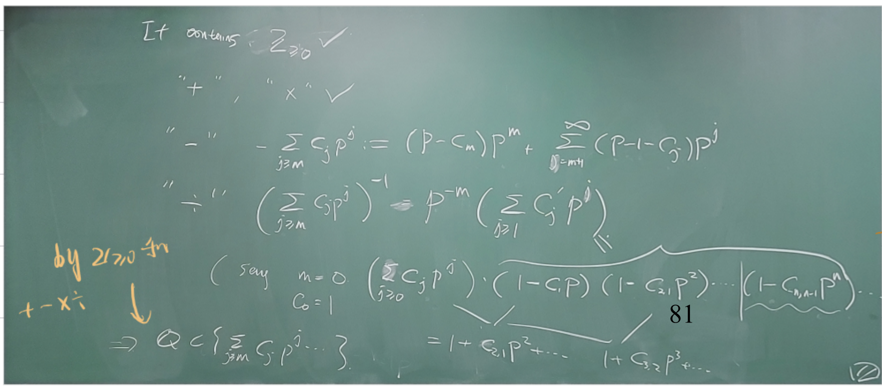
It remains to show that $\mathbb{Q}_p$ is a field $\boxed{\text{containing } \mathbb{Q}}$.

It contains $\mathbb{Z}_{\ge 0}$ ✓

"+" ✓ "×" ✓

"−" : $-\sum_{j \ge m} c_j p^j \overset{def}{=} (p - c_m) p^m + \sum_{j=m+1}^{\infty} (p-1-c_j) p^j$

"÷": $(\sum_{j \ge m} c_j p^j)^{-1} = p^{-m} (\sum_{j \ge 1} c_j' p^j)$



the only completion besides $\mathbb{R}$

Moreover ①: $|\gamma|_p = p^{-k}$, discreate, $\Rightarrow$ each ball $B(\overset{\text{radius}}{r}, \overset{\text{center}}{x})$ is both open and closed.

quite important later.

②: Since $|x-y|_p \le \max\{|x|_p, |y|_p\}$, we have

$|x-y|_p < r$, $|y-z|_p < r$, then $|x-z| < r$.

$\Rightarrow$ Every element in a ball is the center of this ball.

(First $B(r,y) \subset B(r,x)$, $\forall x \in B(r,y)$, then $B(r,x) \subset B(r,y)$ 对称 $\Rightarrow$ $B(r,x) = B(r,y)$)

Corollary: Every 2 balls are either disjoint or one contains another.

③ Every $B(p^m, x)$ contains exactly p^{m-n} balls of radius p^n $(n<m)$

$$B(p^m,x) = x + \sum_{j \ge -m} c_j p^j \qquad \text{something} + \sum_{j \ge -n} c_j p^j$$

$$= \underbrace{x + \sum_{j=-m}^{-n+1} c_j p^j}_{p^{m-n} \text{ options}} + \sum_{j \ge -n} c_j p^j$$

③ $\Rightarrow$ Every bounded sequence has a convergent subsequence ↗ sequentially compact.

$\Rightarrow$ $\mathbb{Q}_p$ is locally compact (参看度量空间, compactness 多 criteria 等价)

④ $(\mathbb{Q}_p, +)$, $(\mathbb{Q}_p \setminus \{0\}, \times)$, locally compact topological group.

For its Haar measure on $(\mathbb{Q}_p, +)$, say $\overset{\text{Haar measure}}{\lambda}(\underset{\text{the integer ring}}{B(0,1)}) = 1$, then every ball of radius p^m has

measure measure p^m, then by outer regularity

$$\lambda(E) = \inf\{\sum p^{m_j} : E \subset \bigcup_1^\infty B(p^{m_j}, x_j)\}$$

Section 2.4: The modular functions

let λ be left (WLOG) Haar measure on G: $\begin{cases} \lambda(xE) = \lambda(E) \\ \int L_y f \, d\lambda = \int f \, d\lambda \end{cases}$ (L_y: left translation by y)

$$\int L_y f \, d\lambda = \underbrace{\int f \, d((L_y)_* \lambda)}_{\text{measure}} = \int f(y^{-1}x) \, d\lambda(x)$$

Question: What is $\int f(xy) \, d\lambda(x)$?

$$= \int R_y f \, d\lambda = \int f \, d((R_y)_* \lambda)$$

For each y, $\int f(zxy) \, d\lambda(x) = \int f(xy) \, d\lambda(x)$

$(R_y)_* \lambda$ is left-invariant

now by uniqueness, $(R_y)_* \lambda = \underbrace{\Delta(y^{-1})}_{\text{constant}} \lambda$

In fact $\Delta(y^{-1})$ is independent in the choice of λ

$$\boxed{\int f(xy)\, d\lambda(x) = \Delta(y^{-1}) \int f(x)\, dx}$$

To see this, fix λ_0, then $\forall \lambda$, $\lambda = c_\lambda \cdot \lambda_0$. Say $\Delta(y^{-1})$ is given by λ_0, then $(R_y)_* \lambda = c_\lambda (R_y)_* \lambda_0$

$$= \underbrace{c_{\lambda_0} \cdot \Delta(y^{-1})\, \lambda_0}$$

$$= \Delta(y^{-1}) \cdot \lambda$$

Prop 2.4: $\overset{\text{the modular function of } G}{\Delta}$ is a continuous homomorphism from G to $\mathbb{R}^\times = (\mathbb{R}_+, \times)$

Proof: $\Delta(1) = 1$, $\Delta(yz) \int f(x)\, d\lambda(x)$

$$= \int \underbrace{f(xz^{-1}y^{-1})}_{= F(xz^{-1})} d\lambda(x) \text{, denote } F(x) = f(xy^{-1})$$

$$= \Delta(z) \int F(x)\, d\lambda(x)$$

$$= \Delta(z) \cdot \Delta(y) \int f(x)\, d\lambda(x) \quad \Rightarrow \Delta(yz) = \Delta(y) \cdot \Delta(z) \text{, so group homomorphism} \rightarrow \Delta(x^{-1}) = \Delta(x)^{-1}$$

For continuity, recall $\forall f \in C_c(G)$, then $\|R_y f - R_{y_0} f\| \to 0$, uniformly as $y \to y_0$

$$\Rightarrow \underbrace{\int R_y f(x)\, d\mu(x)}_{= \Delta(y^{-1}) \int f} \rightarrow \underbrace{\int R_{y_0} f(x)\, d\mu(x)}_{= \Delta(y_0^{-1}) \int f} \quad \square$$

def: G is called uni-modular if $\Delta(x) \equiv 1$, in which case, left Haar is also right Haar.

e.g. Abelian group is unimodular. ✓

Prop 2.27: If K is a compact, then $\Delta|_K \equiv 1$

proof: $\Delta(K)$ is a compact subgroup of $\mathbb{R}^\times \Rightarrow \Delta(K) = \{1\}$. $\square$

Corollary: Every compact group is unimodular.

Prop 2.29: $\overset{\text{denoted by}}{[G, G]} \overset{\text{def}}{=} \{xyx^{-1}y^{-1} : x, y \in G\}$

If $G/[G,G]$ is compact, then G is uni-modular.

Proof: $\Delta(xyx^{-1}y^{-1}) = 1$, then Δ on G induces a Δ on $G/[G:G]$, which must be 1. $\square$

Recall that λ left Haar $\Leftrightarrow$ ρ right Haar

$$\rho(E) = \lambda(E^{-1}), \quad \int f(x)\, d\rho(x) = \int f(x^{-1})\, d\lambda(x)$$

Prop 2.31: $\int f(x)\, d\rho(x) = \int f(x) \boxed{\Delta(x^{-1})\, d\lambda(x)}$

i.e. $d\lambda(-x) = \Delta(x^{-1})\, d\lambda(x)$

Proof: We first show that $\Delta(x^{-1})\, d\lambda(x)$ is right-invariant

$$\int R_y f(x) \underbrace{\Delta(x^{-1})}_{= \Delta(x)^{-1} = \Delta(xy y^{-1})^{-1}} d\lambda(x) = \underbrace{\int R_y (f(x) \cdot \Delta(yx^{-1}))\, d\lambda(x)}_{= \Delta(y^{-1}) \int f(x) \Delta(yx^{-1})\, d\lambda(x)} = \int f(x) \Delta(x^{-1})\, d\lambda(x)$$

then $\Delta(x^{-1})\, d\lambda(x) = c \cdot d\rho(x)$

$\forall\, U \subseteq G$, symmetric compact neighborhood of $1 \Rightarrow \lambda(U) = \rho(U) > 0$

$\Rightarrow (c-1)\lambda(U) = c\rho(U) - \lambda(U) = \int_U \underbrace{(c\Delta(x^{-1}) - 1)}\, d\lambda(x) \quad \leq \varepsilon \lambda(U)$

$\leq \varepsilon$, when U is "small enough"

as Δ is continuous.

$\Rightarrow c = 1.$ $\square$

Remark: If G is not unimodular, Δ is not bounded

so $f(x) \mapsto f(x^{-1})$ is not isometry in $L^p(\lambda)$ $(1 \leq p < \infty)$

However, now we have 2 ways to construct isometries between $L^p(\lambda)$ and $L^p(\rho)$

$L^p(\lambda)$: (1) $f(x) \mapsto f(x^{-1})$ $L^p(\rho)$ 可拓的isometry

$(\int f(x^{-1})\, d\rho(x) = \int f(x)\, d\lambda(x))$

(2) $f(x) \mapsto \Delta(x)^{\frac{1}{p}} f(x)$

$\int \Delta(x) \cdot |f(x)|^p d\rho(x) \overset{\text{prop 2.3)}}{=} \int \Delta(x) |f(x)|^p \Delta(x^{-1})\, d\lambda(x) = \int |f(x)|^p\, d\lambda(x)$

by combining (1), (2), we obtain an isometry on $L^p(\lambda)$

$f(x) \to \Delta(x^{-1})^{\frac{1}{p}} f(x^{-1})$

In particular, $\int f(x)\, d\lambda(x) = \int f(x^{-1}) \Delta(x^{-1})\, d\lambda(x)$, p取1

Lecture 15: Convolutions on G, Homogeneous spaces

Lecture 15-2023 年 4 月 20 日今天潮湿闷热 ☁ 24°C-28°C

主要内容：

- 主要把之前实分析中关于卷积的结论推广到 Locally Compact Group G 上，其中如果 G 有 Unimodular 性质，$f * g$ 和 $g * f$ 有更相近的性质。
- 第二部分简单介绍了 Homogeneous Space（在 Quotient space 上考虑）

其他信息：全程课程录音导致 PDF 文件体积增长迅速，但如果使用 LaTeX 的`\includepdf`貌似可以去除 PDF 中隐含的音频文件。

We finish chapter 2 (Locally compact groups), but will not cover everything

since we mainly deal with abelian group with good properties.

2.5: Convolutions

(G locally compact, λ left Haar measure), dx for convenience

Recall in Real analysis: $\forall f, g \in L^1(\mathbb{R}^d)$, • $f*g = \int f(y)\, g(x-y)\, dy \in L^1$

$\overset{\|}{g*f}$. If take $g = \phi_\varepsilon$ (approximate identity), then $f*\phi_\varepsilon \to f$, as $\varepsilon \to 0$

• $\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$ $\quad$ $f \in L^p$, $g \in L^{p'}$, then $f*g$ is continuous.

$\Rightarrow$

$\|f*g\|_{L^\infty} \le \|f\|_{L^p} \|g\|_{L^{p'}}$

Now in locally compact group

Def: $\forall f, g \in L^1(G)$, def

$$f*g(x) = \int f(y)\, g(y^{-1}x)\, dy \in L^1,$$

$$\forall \phi \in C_c(G). \int \phi \cdot f*g = \iint \phi(x) f(y)\, g(y^{-1}x)\, dx\, dy$$

left invariant

$$= \iint \phi(yx) f(y)\, g(x)\, dx dy$$

However, in general, G $f*g \ne g*f$

LHS: $\int f(y)\, g(y^{-1}x)\, dy$ $\qquad$ RHS: $\int g(y) f(y^{-1}x)\, dy$

$= \int f(xy)\, g(y^{-1})\, dy$ $\qquad$ "=" if G is abelian

$= \int f(xy^{-1})\, g(y) \Delta(y^{-1})\, dy$ $\qquad$ "$\ne$" even if G is unimodular

Observation: $L_z(f*g)(x) = \int f(y)\, g(y^{-1}zx)\, dy = \int f(zy)\, g(y^{-1}x)\, dy$

$= (L_z f)*g$

also $R_z(f*g)(x) = \int f(y)\, g(y^{-1}xz)\, dy = f*(R_z g)$

Prop 2.40 (some details may be omitted). $1 \le p \le \infty$, $f \in L^1$, $g \in L^p$, then

$\boxed{(a)}$ $f*g \in L^p$, $\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$

(b) If G is unimodular, (a) holds for $g*f$

(c) If G is not-unimodular, then (a) holds for $g*f$, if f has compact support.

Proof: (a) $\|f*g\|_{L^p} = \left(\int \left|\int f(y) g(y^{-1}x) dy\right|^p dx\right)^{\frac{1}{p}}$ $\quad$ $\| \|\cdot\|_{L^1} \|_{L^p}$

$\overset{\text{Minkowski}}{\le} \int \left(\int |f(y)|^p |g(y^{-1}x)|^p dx\right)^{\frac{1}{p}}$

$= \|f\|_{L^1} \cdot \|g\|_{L^p}$

(b) Similar by Minkowski, $g*f = \int g(y)\, f(y^{-1}x)\, dy = \int g(xy^{-1})\, f(y)\, \underset{1}{\underset{\|}{\Delta(y^{-1})}}\, dy$, then Minkowski

(c): $|\Delta(y)| \lesssim 1$, $y \in \operatorname{supp} f$.

Prop 2.4.1: Suppose G is unimodular, $f \in L^p$, $g \in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$, then

$f*g \in \underbrace{C_0(G)}_{\text{vanish at } \infty}$, and $\|f*g\|_{\sup} \le \|f\|_{L^p} \|g\|_{L^q}$

$\forall \varepsilon, \exists$ compact $K \subset G$ s.t. $|f|<\varepsilon$. outside K

Proof: ① $\|f*g\|_\infty \le \|f\|_{L^p} \cdot \|g\|_{L^q}$ by Hölder

② Approximate f, g by C_c functions. $\square$

Recall in $\mathbb{R}^n$, $\|f(x+\cdot)-f(\cdot)\|_{L^p} \to 0$, as $x \to 0$, if $f \in L^p$ (L^p-continuity)

In G.

Prop 2.4.2: $1 \le p < \infty$, $f \in L^p(G)$, then $\|L_y f - f\|_{L^p} \to 0$ also by uniformly continuity.

approximate f, g by C_c functions $\|R_y f - f\|_{L^p} \to 0$, as $y \to 1$

Prop 2.4.3: $f \in L^1$, $g \in L^\infty$, then $f*g$ is left uniformly continuous.

$g*f$ is right uniformly continuous.

Proof of 2.4.2: Omit

Proof of 2.4.3: recall $L_y(f*g) = (L_y f)*g$, now

$R_y(g*f) = g*(R_y f)$

$$L_y(f*g)(x) - f*g(x) = (L_y f)*g - f*g(x)$$

$$= (L_y f - f)*g, \quad \text{take } \tilde{f} \in C_c(G), \text{ s.t. } \|f-\tilde{f}\|_{L^1} < \varepsilon$$

$$= (L_y \tilde{f} - \tilde{f})*g(x) + O(\varepsilon)$$

$$\le \|L_y \tilde{f} - \tilde{f}\|_{L^1} + O(\varepsilon) = O(\varepsilon). \quad \square$$

R_y of the similar principle.

When G is discrete, $\delta(x) = \begin{cases} 1, & x=0 \\ 0, & \text{elsewhere} \end{cases} \in C_c(G)$, and $f*\delta(x) = f(x)$

$f*\delta(x) = \int f(y)\,\delta(y^{-1}x)\,dy$

Haar measure on discrete set $\Rightarrow$ counting measure (up to a constant)

For G, general group, a function δ s.t. $f*\delta = f$ might NOT exist!

the following prop (approximating identity)

Prop 2.4.4 (Approximate identity)

Let $\mathcal{U}$ be a neighborhood base at 1. For each $U \in \mathcal{U}$, let ψ_U be a L^1-function s.t.

(i) supp ψ_U is compact

(ii) $\psi_U \ge 0$, and $\int \psi_U = 1$

then $\|\psi_U * f - f\|_{L^p} \to 0$, as $U \to \{1\}$, if $f \in L^p$, $1 \le p < \infty$, or f right uniformly continuous, $p=\infty$

If, in addition, $\psi_U(x^{-1}) = \psi_U(x)$, then the above holds for $\| \underline{f * \psi_U} - f \|_{L^P}$ - - - $\nearrow$ 换序了!

Proof: $\psi_U * f(x) - f(x) = \int \psi_U(y) \underbrace{f(y^{-1}x)}_{= L_{y^{-1}} f(x)} dy \; - \underbrace{f(x)}_{= \int \psi_U(y) f(x) dy}$

$$= \int \psi_U(y) \left(L_{y^{-1}} f(x) - f(x) \right) dy \text{ , then}$$

$$\| \psi_U * f - f \|_{L^P} = \left(\int \left| \int \psi_U(y) (L_{y^{-1}} f(x) - f(x)) dy \right|^p dx \right)^{\frac{1}{p}}$$

$$\overset{\text{minkowski}}{\leq} \int \underbrace{\| L_{y^{-1}} f(x) - f(x) \|_{L^P}}_{\to 0, \text{ as } y \to 1} \psi_U(y) dy$$

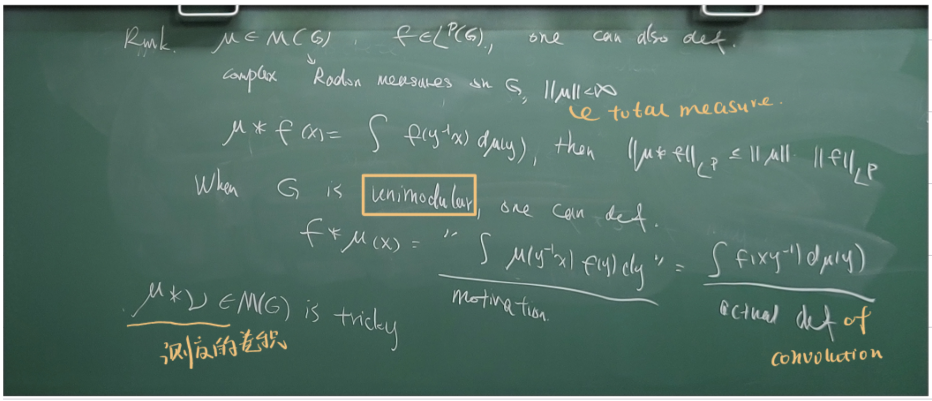
$$\leq \varepsilon \cdot \int \psi_U(y) dy = \varepsilon . \quad \text{as } U \to \{1\}.$$

If $\psi_U(x^{-1}) = \psi_U(x)$, $\underbrace{f * \psi_U(x)}_{\text{换序}} - f(x)$

$$= \int f(y) \underbrace{\psi_U(y^{-1}x)}_{= \psi_U(x^{-1}y)} dy - f(x) = \int \psi_U(y) \left(R_y f(x) - f(x) \right) dy$$

then apply Minkowski $\square$

Remark:



Section 2.6: Homogeneous Spaces

H: closed subspaces , $G \curvearrowright G/H$ homogeneous space , $\underline{G\text{-space}}$ ($\nearrow$ group action)

A space S equipped with an action of G

Model $: G \curvearrowright \underbrace{S}_{\text{locally compact Hausdorff}}$
(G-space)

fix $s_0 \in S$, and let $\underline{H \overset{\text{def}}{=} \{ x \in G : x s_0 = s_0 \}}$ closed

then consider G/H, and when the action is transitive $\forall s_0, s_1 \in S$, $\exists\, x \in G$, $x s_0 = s_1$

then $\Phi : G/H \to S$ is a continuous bijection.

It may $\boxed{\text{not}}$ be a homeomorphism, $\underbrace{\mathbb{R}}_{\text{discrete topology}} \curvearrowright \underbrace{\mathbb{R}}_{\text{regular topology}}$

Prop 2.46 : Φ is a homeomorphism when $\underline{G \text{ is } \sigma\text{-compact}}$.

$S \cong G/H$ 同胚

skip the proof.

Goal: 有闭子群上的积分, S上的积分 $\Rightarrow$ 表达出 G上的积分

Def: $\forall f \in C_c(G)$, define $\overset{\text{projection}}{Pf}(xH) \overset{\text{def}}{=} \boxed{\int_H f(x\xi)\,d\xi \in C_c(G/H)}$

main theorem

Thm 2.5.1: $\exists$ G-invariant measure μ on $G/H \Leftrightarrow \Delta_G|_H = \Delta_H$,

Moreover, in this case μ is unique up to a constant factor, and if this factor is suitably chosen, we have

$$(*)\quad \int_G f(x)\,dx = \int_{G/H} Pf\,d\mu = \int_{G/H}\int_H f(x\xi)\,d\xi\,d\mu(xH), \quad \forall f \in C_c(G)$$

Remark: ①: It holds when H is compact

②: It holds for $f \in L^1$, or $f \geq 0$, supp f is σ-finite (by approximation of cpt-supp func (e.g. in measure theory))

③: It can be reduced to "Fubini" when G is second countable (see Notes and preferences.)

proof: the key of the proof is to show that all $C_c(G/H)$ function can be written as Pf, for some $f \in C_c(G)$ (compact!)

提升集合

Lemma 2.4.8: If $E \subset G/H$ is compact, $\exists K \subset G$, compact. s.t. $q(K) = E$ (canonical quotient map: $G \to G/H$)

proof: Take a compact neighborhood V of 1 in G, then $E \subset \bigcup_{x \in q^{-1}(E)} q(xV)$, open cover

$\Rightarrow E \subset \bigcup_{j=1}^{n} q(x_j V)$ by locally compact, finite open cover

take $K = q^{-1}(E) \cap (\bigcup_{j=1}^{n} x_j \bar{V})$. ▯

提升函数(indicator function)

Lemma 2.49: If $F \subset G/H$ compact, $\exists f \in C_c(G)$, $f \geq 0$, s.t. $Pf|_F = 1$

proof: take $\phi \in C_c(G/H)$, $\phi = 1$ on F, and $g \in C_c(K)$, where K is the compact set from Lemma 2.48 ($q(K)=F$)

let $f = \dfrac{\phi \circ q}{Pg \circ q}\, g$, then $\forall xH \in F$,

$$Pf(xH) = \int_H f(x\xi)\,d\xi = \frac{\phi(xH)}{Pg(xH)} \underbrace{\int_H g(x\xi)\,d\xi}_{= Pg(xH)} = \phi(xH)$$

▯

Proposition 2.50: If $\phi \in C_c(G/H)$, then $\exists f \in C_c(G)$ s.t. $Pf = \phi$, $q(\text{supp} f) = \text{supp}(\phi)$, and $f \geq 0$, if $\phi \geq 0$

proof: $f = (\phi \circ q) \cdot \underbrace{g}_{\text{from lemma 2.49}}$

then $Pf = \phi \cdot \underbrace{Pg}_{=1 \text{ on supp } \phi}$, other properties of f are obvious. ▯

Now. proof of thm 2.5.1

Proof: Suppose $\exists$ a G-invariant measure μ on G. define $f \mapsto \int Pf(xH)\,d\mu(xH)$ $\forall f \in C_c(G)$

left-invariant: $L_y f \mapsto \int Pf(yxH)\,d\mu(xH) = \int Pf(xH)\,d\mu(xH)$

by uniqueness of Haar measure $\int f\,dx = c\cdot \int Pf\,d\mu$ $\to$ uniquely determined by lemma 2.5, one we choose c

one can take $c=1$, $\int f\,dx = \int Pf\,d\mu$

$$= \int_{G/H} \int_H f(x\xi)\,d\xi\, d\mu(xH)$$

Then $\Delta_G(\eta) \int_G f(x)\,dx \overset{\text{by def of } \Delta}{=} \int_G f(x\eta^{-1})\,dx, \ \forall \eta \in H$

$$= \int_{G/H} \underbrace{\int_H f(x\xi\eta^{-1})\,d\xi}\, d\mu$$

$$= \Delta_H(\eta) \underbrace{\int_{G/H} \int_H f(x\xi)\,d\mu}_{= \int_G f} \Rightarrow \Delta_G(\eta) = \Delta_H(\eta), \ \forall \eta \in H.$$

Conversely $\Leftarrow$: assume $\Delta_G|_H = \Delta_H$, We need to define a positive linear functional on G/H

Def: we have proved every $C_c(G/H)$ function can be written as Pf

we would like to define $Pf \longmapsto \int_G f\,dx$ on $C_c(G/H)$, $f \in C_c(G)$

G invariant ✓ positive ✓

It remains to show that it is well-defined, i.e. $\underset{\int_H f(x\xi)\,d\xi}{Pf} = 0 \Rightarrow \int_G f = 0$, $f \in C_c(G)$

By Lemma 2.49, $\exists\, \phi \in C_c(G)$ s.t. $P\phi = 1$ on $\underbrace{q(\operatorname{supp} f)}_{\text{cpt}}$ $\Rightarrow \int_H \phi(x\xi)\,d\xi = 1$ on supp f

$$0 = \int_G \phi(x) \int_H f(x\xi)\,d\xi\,dx \overset{\text{Fubini}}{=} \int_H \int_G \phi(x) f(x\xi)\,dx\,d\xi$$

$$= \int_H \Delta_G(\xi^{-1}) \int_G \phi(x\xi^{-1}) f(x)\,dx\,d\xi$$

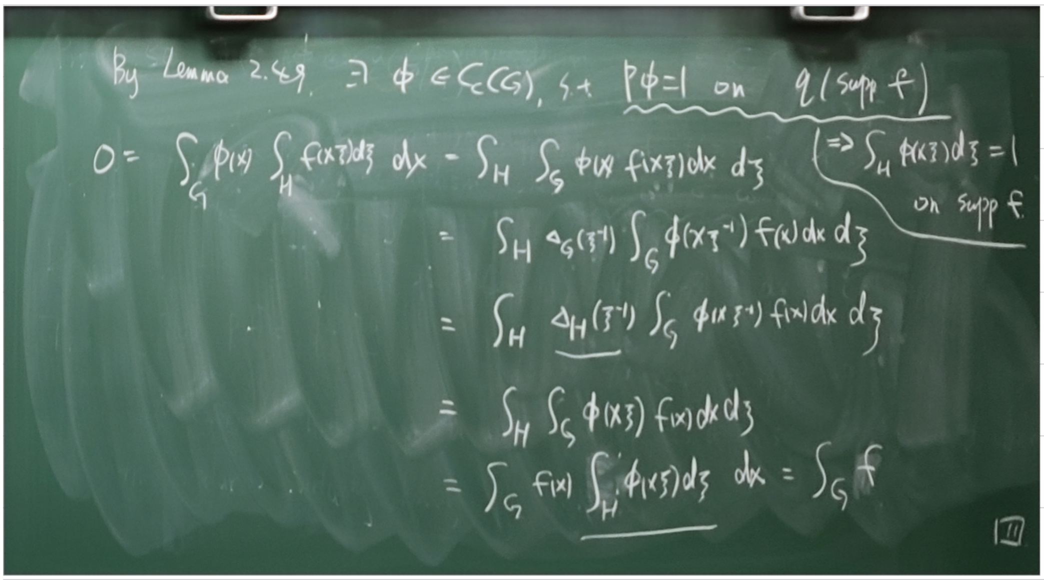
$\parallel$ by condition

$$= \int_H \Delta_H(\xi^{-1}) \int_G \phi(x\xi^{-1}) f(x)\,dx\,d\xi$$

$$= \int_H \underline{\int_G \phi(x\xi)} f(x)\,dx\,d\xi$$

$$= \int_G f(x) \int_H \phi(x\xi)\,d\xi\,dx = \int_G f(x)\,dx.$$

$\boxed{///}$



Lecture 16: Banach Algebra and Basic Representation Theory

Lecture 16-2023 年 4 月 25 日今天降温 ☁ 18°C-23°C

主要内容： Proposition 1.27

其他信息：

partial collection of chapter 1 and chapter 3 $\Rightarrow$ "五一"后直接 chapter 4 (locally compact abelian group上的 Fourier 分析)

如果需要更复杂的结论

再补!

rather deep theory

Chapter 1. Banach Algebra and spectrum theorem

Def: Banach algebra $\mathcal{A}$ over $\mathbb{C}$ is an algebra with a norm $\|\cdot\|$ that makes it a Banach space with $\|x\cdot y\| \le \|x\|\cdot\|y\|$.

- Unital Banach algebra: $\exists\, e$

e.g. 矩阵转置, transpose

- Involution on $\mathcal{A}$: an automorphism $\mathcal{A}\to\mathcal{A}$ such that

$$(\lambda x)^* = \bar{\lambda} x^*, \quad x^{**} = x$$

$$(x+y)^* = x^* + y^*, \quad (x\cdot y)^* = y^*\cdot x^*$$

(Banach) $*$-algebra = Banach algebra + involution

C^*-algebra = $*$-algebra + $\|x\cdot x^*\| = \|x\|^2$

$\Rightarrow \|x\| = \|x^*\|$, for $\|x\|^2 = \|x\cdot x^*\| \le \|x\|\cdot\|x^*\|$

$\|x^*\|^2 = \|x^*\cdot (x^*)^*\| = \|x^*\cdot x\| \le \|x\|\cdot\|x^*\|$

Homomorphism: same in algebra

$*$-Homomorphism: Homomorphism + $\phi(x^*) = \phi(x)^*$

Example: X compact Hausdorff

有identity

- $C(X)$ is unital C^*-algebra, $f\cdot g$. $e = 1_X$, $\|f\| = \sup|f|$

$$f^* = \bar{f}, \quad f * f^* = |f|^2$$

- If X is not compact, then $C(X) \overset{\text{def}}{=} \{f \text{ continuous}. \|f\|_{\sup} < \infty\}$, $C(X)$ is still unital

but $C_0(X)$ is not unital.

vanishing at ∞

Example 2: H Hilbert space

$\mathcal{L}(H) = \{\text{bounded linear operators on } H\}$, unital C^*-algebra

Example 3: $L^1(G)$: $*$-algebra, not C^*. $\|f*g\|_{L^1} \overset{\le}{\neq} \|f\|_{L^1}\cdot\|g\|_{L^1}$

unital if and only if G is discrete.

$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ to ensure $(f*g)^* = g^* * f^*$

From non-unital to unital. suppose $\mathcal{A}$ is non-unital

Construct $\widetilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$, with $(x,a)\cdot(y,b) = (xy + ay + bx, ab)$, and

$$\|(x,a)\| = \|x\| + |a|$$

In this case, $e = (0,1)$, and $\mathcal{A} \cong \mathcal{A} \times \{0\}$ is a closed (maximal) ideal of $\widetilde{\mathcal{A}}$

If $\mathcal{A}$ is $*$-algebra, so is $\widetilde{\mathcal{A}}$ with $(x,a)^* = (x^*, \bar{a})$.

Example 1:

补充的 identity → unital

total variation.

$\mathcal{A} = L^1(\mathbb{R})$, $\widetilde{\mathcal{A}} = \operatorname{Span}\{\mathcal{A}, \delta\} \subseteq M(G) = \{\mu\}$, $\|\mu\| = |\mu|(\mathbb{R})$

finite Borel measure.

$(f, a) = f + a\delta$, $\|(f,a)\| = \|f\|_{L^1} + |a|$

L^1 $\mathbb{C}$

Example 2:

$C_0(X)$, X is not compact. consider $\widetilde{\mathcal{A}} = \underbrace{\operatorname{Span}\{\mathcal{A}, 1_X\}}_{\text{Indeed a unital algebra } = C(\hat{X})} \subset C(X)$

one-pt compactification!

现在，范数的选取上会有一点问题！

参考 Example 1 above

$\forall f \in \widetilde{\mathcal{A}}$, $\|f\| = \|f - f(\infty)\|_{\sup} + |f(\infty)| \neq \|f\|_{C(X)}$

不等于 Embed 空间的 norm! ⇒ Not a severe problem

Example 1 的选取的 norm 不够好

⇓

the following prop

So $\mathcal{A}$ is $C^* \not\Rightarrow$ so is $\widetilde{\mathcal{A}}$ ($\|x\cdot x^*\| = \|x\|^2$)

Proposition 1.27: If $\mathcal{A}$ is a non-unital C^*-algebra, $\exists!$ norm on $\widetilde{\mathcal{A}}$ that makes $\widetilde{\mathcal{A}}$ a C^*-algebra

and this norm agrees with the original norm on $\mathcal{A}$.

a little complicated.

proof: define $\|(x,a)\| \overset{\text{def}}{=} \sup\{\underbrace{\|(x,a)\cdot(y,0)\|}_{=\|xy+ay\|} : y \in \mathcal{A}, \|y\| \le 1\}$. ⋯ ▨

Now $\mathcal{A}$ is unital, then we can discuss x^{-1}

Simple facts: $\|x\| < 1 \Rightarrow (e - x)^{-1} = \sum_{n=0}^{\infty} x^n$

Corollary 1. If $|\lambda| > \|x\|$, then $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n$ (rescaling)

2. $\|y\| \cdot \|x^{-1}\| < 1 \Rightarrow (x - y)^{-1} = x^{-1} \sum_{n=0}^{\infty} (yx^{-1})^n$

$= x^{-1}(e - yx^{-1})^{-1}$

(取逆 being continuous) ($y \mapsto \delta y \Rightarrow$ continuous)

3. $\|y\| \cdot \|x^{-1}\| \le \frac{1}{2} \Rightarrow \|(x-y)^{-1} - x^{-1}\|$

$= $ by (2). $\| x^{-1} \cdot \sum_{n=1}^{\infty} (yx^{-1})^n \| = \| \underbrace{yx^{-2} \sum_{n=0}^{\infty} (yx^{-1})^n}_{\text{bounded}} \|$

$\to 0$, as $\|y\| \to 0$

(1), (2), (3) $\Rightarrow \{x \text{ is invertible}\}$ is open, and $x \mapsto x^{-1}$ is continuous. (3)

Def: $\forall x \in \mathcal{A}$ unital, the spectrum of x is defined by $\sigma(x) = \{\lambda : \lambda e - x \text{ is not invertible}\}$

closed (∵之前 invertible 是 open) and $\lambda \to \lambda e - x \in \mathcal{A}$ is continuous.

Also $\sigma(x) \subset B_{\|x\|}$.

Def: For $\lambda \notin \sigma(x)$, $R(\lambda) \overset{def}{=} (\lambda e - x)^{-1}$ is called the resolution element of x

Lemma 1.5: $R(\lambda)$ is $\boxed{\text{analytic}}$ in $\mathbb{C} \setminus \sigma(x)$

$R'(\lambda)$ exists, or $\forall \phi \in \mathcal{A}^*$ (bounded linear functional), $\phi \circ R(\lambda)$ is analytic.

Proof: $\forall \lambda, \mu \notin \sigma(x)$

Calculate $\boxed{\lim_{\mu \to \lambda} \frac{R(\mu) - R(\lambda)}{\mu - \lambda}}$, $(\mu - \lambda) e = (\mu e - x) - (\lambda e - x)$

$= \underbrace{(\lambda e - x) R(\lambda)}_{e} (\mu e - x) - (\lambda e - x) R(\mu) (\mu e - x)$

$= (\lambda e - x)(R(\lambda) - R(\mu))(\mu e - x)$

$\Rightarrow \frac{R(\lambda) - R(\mu)}{\mu - \lambda} = -R(\lambda) R(\mu) \Rightarrow R'(\lambda) = -R(\lambda)^2$.

∎

Proposition 1.6: $\sigma(x)$ is $\boxed{\text{non-empty}}$, $\forall x$

Proof: $R(\lambda) \to 0$, as $|\lambda| \to \infty$. So if $\sigma(x) = \phi$, then $\phi \circ (R(\lambda))$ is bounded analytic

$\Rightarrow \phi \circ (R(\lambda)) = \text{constant} = 0$.

Contradiction as $\forall y, \exists \phi$ s.t. $\phi(y) \ne 0$, by e.g. Hahn-Banach

∎

Theorem 1.7: If every non-zero element in $\mathcal{A}$ is invertible, then $\mathcal{A} \cong \mathbb{C}$

proof: by prop 1.6, $\forall x \in \mathcal{A}, \exists \lambda \in \mathbb{C}$, s.t. $\boxed{\lambda e - x \text{ is not invertible}}$

by our assumption, $\forall$ non-zero element is invertible $\Rightarrow \lambda e = x$

$\Rightarrow \mathcal{A} = \mathbb{C} \cdot e$. $\mathcal{A} \cong \mathbb{C}$ ∎

Def: the spectral radius of x is $\rho(x) \overset{def}{=} \sup\{|\lambda| : \lambda \in \sigma(x)\}$

Thm 1.8: $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$

Application ① $f \in \underbrace{C(X)}$ → compact , $\|f\|_{\sup} = \lim_{n\to\infty} \|f\|_{L^n}$

对于不同的空间

② $f \in L^1(\mathbb{R}^d)$. $\lim_{k\to\infty} \|\underbrace{f * \cdots * f}_{k}\|_{L^1}^{\frac{1}{k}} = \|\hat{f}\|_{\sup}$

Will be used in chapter 4

The above are mainly all about the Banach algebra in this class.

Chapter 3 is about representation theory (Basic $\underbrace{\text{representation Theory}}_{\text{mainly unitary in this book}}$)

G: locally compact group

$\mathcal{H}_\pi$: Hilbert space

Unitary representation, continuous homomorphism $\pi: \overset{群}{G} \to \underbrace{\overset{也是群}{U(\mathcal{H}_\pi)}}_{\text{unitary operators}}$

$\pi(xy) = \pi(x)\cdot\pi(y)$

$\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, $\dim(\pi) \overset{\text{def}}{=} \dim(\mathcal{H}_\pi)$

Example: $\mathcal{H}_\pi = L^2(G)$, $\pi_{\underset{\text{Left}}{L}}(x) f(y) = f(x^{-1}y)$, then

$\langle \pi_L(x) f, \pi_L(x) g \rangle = \int f(x^{-1}y)\, g(x^{-1}y)\, dy$

$= \langle f, g \rangle$ Left-invariant

Left-regular representation.

$\pi_R(x) f(y) \overset{\text{def}}{=} f(x^{-1}y)\cdot\Delta(x)^{\frac{1}{2}}$, right regular representation

Def: $\pi_1: G \to \mathcal{H}_{\pi_1}$, $\pi_2: G \to \mathcal{H}_{\pi_2}$, we say π_1, π_2 are equivalent, if $\exists$ unitary $U: \mathcal{H}_{\pi_2} \to \mathcal{H}_{\pi_1}$

$\pi_1 \circ U = U \circ \pi_2$

$$\begin{tikzcd}
\mathcal{H}_{\pi_2} \arrow[r, "\pi_2(x)"] \arrow[d, "U"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_{\pi_2} \arrow[d, "U"] \\
\mathcal{H}_{\pi_1} \arrow[r, "\pi_1(x)"] & \mathcal{H}_{\pi_1}
\end{tikzcd}$$

$\forall x$ commutative !

More generally, one can consider

$$\mathcal{C}(\pi_1, \pi_2) \overset{\text{def}}{=} \{ \underset{\text{Bounded linear}}{T: \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}},\ \begin{tikzcd}
\mathcal{H}_{\pi_1} \arrow[r, "\pi_1"] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_{\pi_1} \arrow[d, "T"] \\
\mathcal{H}_{\pi_2} \arrow[r, "\pi_2"] & \mathcal{H}_{\pi_2}
\end{tikzcd}\quad T \circ \pi_1 = \pi_2 \circ T \}$$

and denote $\mathcal{C}(\pi) = \mathcal{C}(\pi, \pi) = \{ T \in B(\mathcal{H}_\pi) : T\circ\pi = \pi\circ T \}$

closed under taking adjoint.

Next class, we will show some results using about definitions. (可见讲稿)

Lecture 17: cont. Basic Representation Theory

Lecture 17-2023 年 4 月 27 日开始升温 ☁ 22°C-28°C

主要内容：首先补充了测度的卷积的相关内容

- Irreducible representation
- **Main Theorem**: If G is abelian, then every irreducible unitary representation is **1-dimensional** ($H_\pi \cong \mathbb{C}$).

- A **function of positive type** on a locally compact group G is a function $\phi \in L^\infty(G)$ that defines a positive linear functional on the Banach $*$-algebra $L^1(G)$, i.e. that satisfies

$$\int (f^* * f)\phi \geq 0, \text{ for all } f \in L^1(G).$$

其他信息：

五一后开始 Abelian group ↗ 很多地方都可以视为 $\mathbb{R}^n$

补充 measure 的 convolution ↗ 可以推广到 distribution

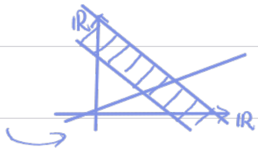
$L^1 \subset \mathcal{M}(G)$, $\boxed{\text{convolution of measures}}$

$\downarrow$

finite Borel measure. $\int_{\mathbb{R}^n} f\, d\mu * \nu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x+y)\, d\mu(x)\, d\nu(y)$

$\underbrace{d\mu * \nu}_{\downarrow}$

$\mu \times \nu \xrightarrow{\pi(x,y)=x+y} \pi_*(\mu\times\nu)$



If μ, ν Radon measure $\not\Rightarrow$ $\mu * \nu$ is also Radon

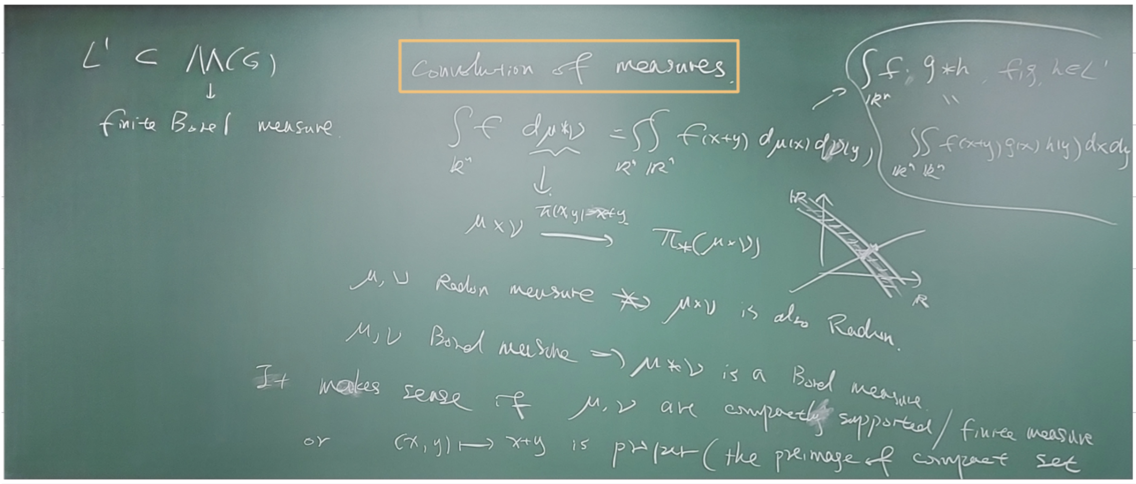
If μ, ν Borel measure $\Rightarrow \mu * \nu$ is a Borel measure

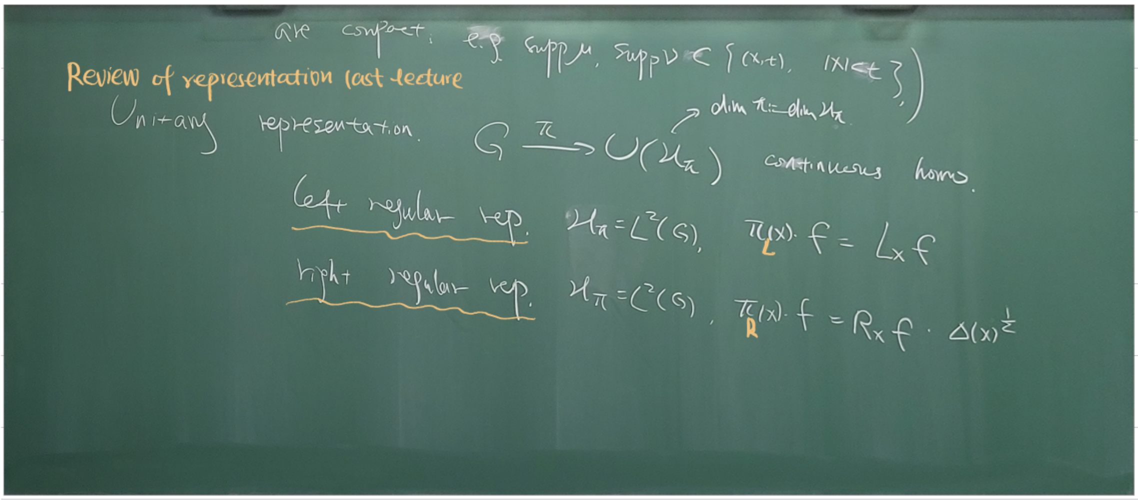
It makes sense if μ, ν are compactly supported/finite measure

or $(x,y) \mapsto (x+y)$ is proper

↑ the pre-image of compact set are compact

e.g. $\operatorname{supp}\mu, \operatorname{supp}\nu \subset \{(x,t),\ |x|<t\}$





Equivalence : $\mathcal{H}_{\pi_1} \xrightarrow{U} \mathcal{H}_{\pi_2}$ U is unitary from $\mathcal{H}_{\pi_1}$ to $\mathcal{H}_{\pi_2}$

$$\begin{tikzcd} \mathcal{H}_{\pi_1} \arrow[r, "U"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_{\pi_2} \arrow[d, "\pi_2"] \\ \mathcal{H}_{\pi_1} \arrow[r, "U"] & \mathcal{H}_{\pi_2} \end{tikzcd}$$

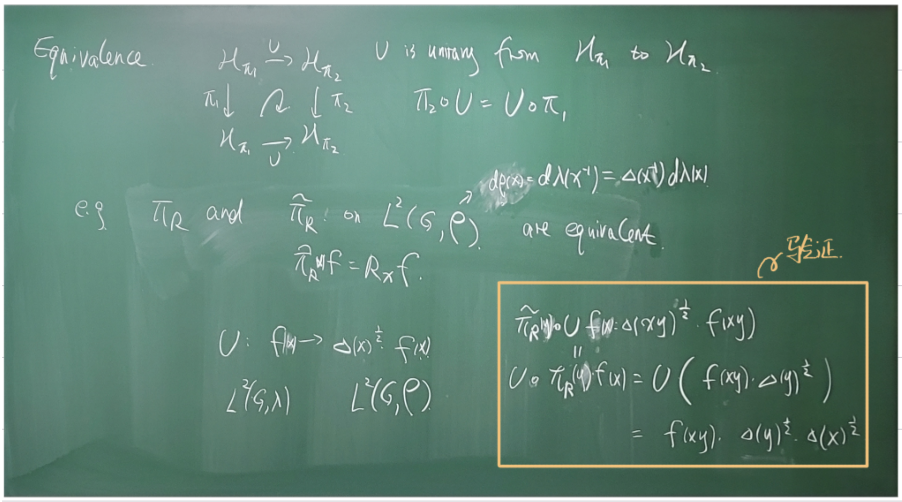
$\pi_2 \circ U = U \circ \pi_1$ (the diagram commutes)

$d\rho(x) = d\lambda(x^{-1}) = \Delta(x^{-1})\, d\lambda(x)$

e.g. π_R and $\tilde{\pi}_R$ on $L^2(G,\rho)$ are equivalent

$\tilde{\pi}_R(x) f = R_x f$

$U:\ \underset{L^2(G,\lambda)}{f} \to \underset{L^2(G,\rho)}{\Delta(x)^{?} f}$



Def: $\overset{C}{\mathcal{C}}(\pi_1,\pi_2) \overset{def}{=} \{ T \in B(H_{\pi_1}, H_{\pi_2}),\ T\circ\pi_1 = \pi_2\circ T,\ \forall x \}$

$\mathcal{C}(\pi,\pi) = \mathcal{C}(\pi)$, $\pi\circ T = T\circ\pi$ (commutant or centralizer of π.)

Def: Say $\mathcal{M}$ is closed subspace of H_π, then $\mathcal{M}$ is called invariant if $\pi(x)\mathcal{M} \subset \mathcal{M}$, $\forall x \in G$, then $\pi|_\mathcal{M}$ is called a subrepresentation, and we call π $\boxed{\text{reducible}}$ if a proper $\mathcal{M}$ exists otherwise we call it irreducible.

Prop 3.1: If $\mathcal{M}$ is invariant under π, then so is $\mathcal{M}^\perp$ $\Rightarrow$ **Cor**: $\pi = \pi|_\mathcal{M} \oplus \pi|_{\mathcal{M}^\perp} = \pi^\mathcal{M} \oplus \pi^{\mathcal{M}^\perp}$

proof: If $u \in \mathcal{M}$, $v \in \mathcal{M}^\perp$, $\forall x \in G$, $\langle \pi(x)v, u\rangle = \langle v, \pi(x^{-1})u\rangle = 0$

so $\pi(x)v \in \mathcal{M}^\perp$ ∎

Remark: For non-unitary representation, it may fails.

Say $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \circlearrowright \mathbb{R}^2$, the only invariant space is span$\{(1,0)\}$

Def: π is called cyclic if $\exists u \in H_\pi$ s.t. $H_\pi = \overline{\{\pi(x)u : x \in G\}} \overset{def}{=} \mathcal{M}_u \to$ Invariant

Prop: Every unitary representation is a direct sum of cyclic representation.

proof: by prop 3.1 and Zorn's lemma contradicting the maximality.

Cor: Irreducible representation must be cyclic.

Prop 3.4: $\mathcal{M}$ is a closed subspace of H_π, and let P be the orthogonal projection $P: H_\pi \to \mathcal{M}$

then $\mathcal{M}$ is invariant if and only if $P \in \mathcal{C}(\pi)$ i.e. $\pi\circ P = P\circ\pi$)

Proof: "$\Leftarrow$": If $\pi\circ P = P\circ\pi$, and $v\in\mathcal{M}$, then $\boxed{\pi(x)v} = \pi(x)Pv = \boxed{P\pi(x)v}\in\mathcal{M}$. So $\mathcal{M}$ is invariant

"$\Rightarrow$" If $\mathcal{M}$ is invariant, we have $\pi(x)Pv = \pi(x)v = P\pi(x)v$, for $v\in\mathcal{M}$.

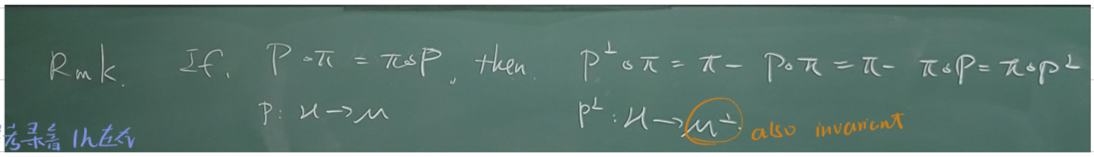
$\mathcal{M}^\perp$ also invariant!

and $\pi(x)Pv = 0 = P\pi(x)v$, for $v\in\mathcal{M}^\perp$ (prop 3.1)

Hence $\pi(x)P = P\pi(x)$. ▨

Remark:

书上没有写 参考录音 1h左右



So "$P\circ\pi = \pi\circ P$" $\Rightarrow$ both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant $\Rightarrow$ $\pi = \pi^{\mathcal{M}}\oplus\pi^{\mathcal{M}^\perp}$

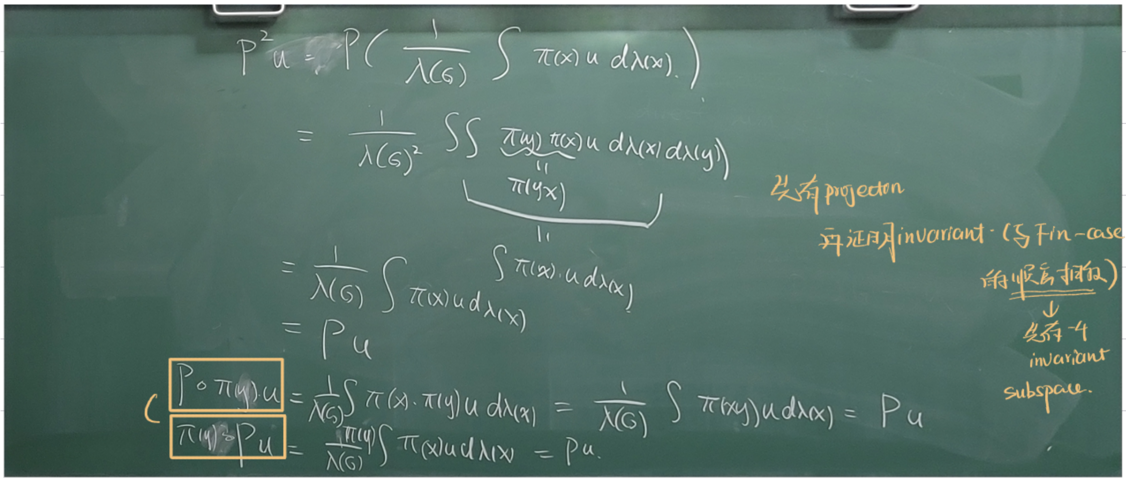
implies thm 3.1

When G is compact (unimodular), one can write $P = \frac{1}{\lambda(G)}\int\pi(x)\,d\lambda(x)$, in particular when G is finite.

有限群求和结果

then $\boxed{P = \frac{1}{|G|}\sum_{x\in G}\pi(x)}$

验证P为一个projection



先证这个再把 Schur's lemma 当作 corollary

Theorem: If G is Abelian, then every irreducible unitary representation is one-dimensional ($\mathcal{H}_\pi\cong\mathbb{C}$)

Proof: It suffices to prove $\pi(x) = c_x\cdot\underbrace{I}_{\text{identity on }\mathcal{H}_\pi}$

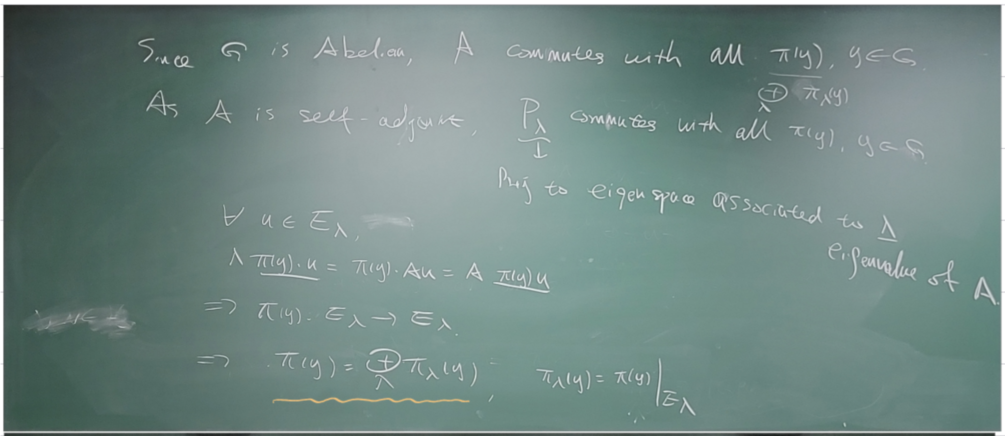
If not, one of $A \overset{\text{def}}{=} \frac{\pi(x)+\pi^*(x)}{2}$, $B \overset{\text{def}}{=} \frac{\pi(x)-\pi^*(x)}{2}$ is not a multiple of I, Say A is not (self-adjoint)

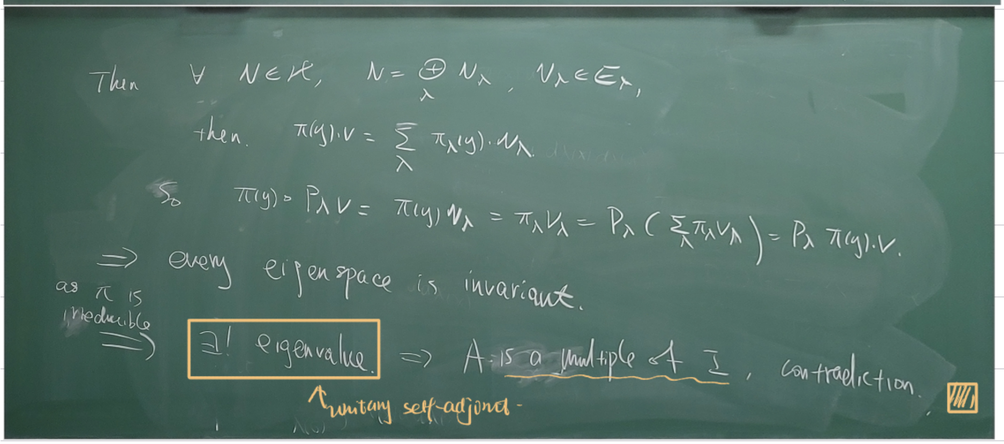
Since G is abelian, A commutes with all $\underset{\oplus_\lambda\pi(y)}{\pi(y)}$, $y\in G$.

As A is self-adjoint. P_λ commutes with all $\pi(y)$, $y\in G$.

P_λ: projection to eigenspace associated to λ (eigenvalue of A)

则行.





the Schur's lemma is more general!

Section 3.3: Functions of positive type $P \subset L^\infty$

Def: ϕ s.t. $\int (f^* * f)\,\phi \geq 0$, $\forall f \in L^1(G)$ $\hat{\mu}$

$\|$ In Euclidean space

$\int |\hat{f}|^2 d\mu$

proved in chapter 4

Bochner thm: When G is abelian $P = \{\hat{\mu} : \mu \in M(G)\}$

finite Borel measure.

will be used later

Prop 3.15: $\boxed{\langle \pi(x)u, u\rangle \in P}$ $\forall u$

proof: $\int f^* * f(x)\, \langle \pi(x)u, u\rangle\, dx$

$= \iint \Delta(y^{-1})\, \overline{f(y^{-1})}\, f(y^{-1}x) \langle \pi(x)u, u\rangle\, dy\, dx$

$= \iint \overline{f(y)}\, f(x) \cdot \langle \underbrace{\pi(y^{-1}x)}_{\pi(y)^*\pi(x)} u, u\rangle\, dy\, dx = \iint \overline{f(y)} \cdot f(x) \langle \pi(x)u, \pi(y)u\rangle\, dy\, dx$

$= \iint \langle f(x)\pi(x)u, f(y)\pi(y)u\rangle\, dy\, dx$

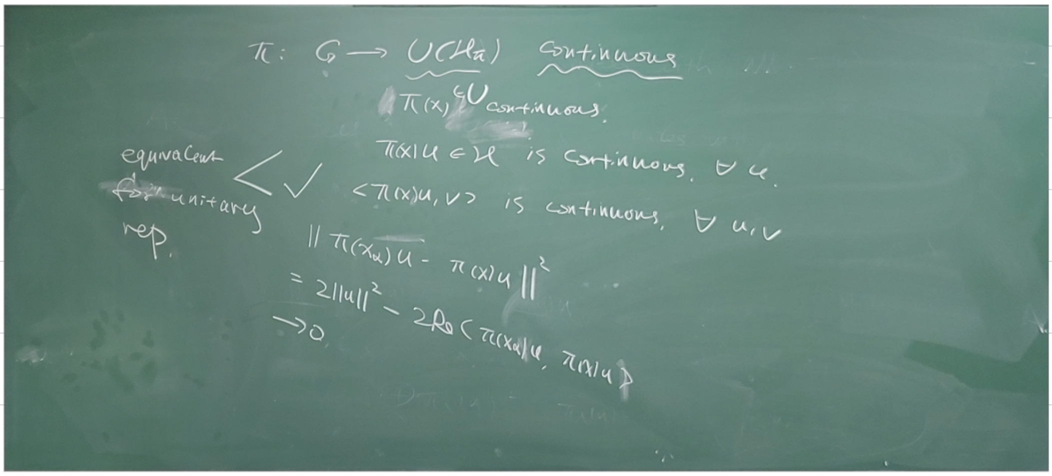
$= \|\pi(f)u\|^2 \geq 0$. ▨

最后考虑一些 topology 相关的问题!

$\pi : G \longrightarrow U(H_\pi)$ continuous — Strongest

$\pi(x) \in U$ continuous

Equivalent for unitary representation $\begin{cases} \pi(x)u \in H \text{ is continuous, } \forall u \\ \langle \pi(x)u, v\rangle \text{ is continuous, } \forall u, v \end{cases}$ 100 ✓ — Weakest


π: G → U(H_π) continuous
π(x) ∈ U continuous.
π(x)u ∈ H is continuous. ∀ u.
equivalent for unitary rep.
<π(x)u, v> is continuous. ∀ u, v
|| π(x_α)u − π(x)u ||²
= 2||u||² − 2Re <π(x_α)u, π(x)u>
→ 0

Lecture 18: Analysis on locally compact abelian groups

Lecture 18-2023 年 5 月 4 日热 ☼ 24°C-30°C

主要内容：介绍了 Dual group $\hat{G}$，并在其上引入了 weak-* topology

其他信息：

补课(上一次)：距离Final还有四周！

Def: π is called cyclic if $\exists\, u \in \mathcal{H}_\pi$ s.t. $\mathcal{H}_\pi = \overline{\{\pi(x)u : x \in G\}}$ ① $\boxed{\text{span}}$ → 这里的norm都不变 ∵ unitary 所以需要span.

$\overset{\text{def}}{=} M_u$ → Invariant

Prop: Every unitary representation is a direct sum of cyclic representation.

proof: by prop 3.1 and Zorn's lemma, contradicting the maximality.

Cor: Irreducible representation must be cyclic.

② For unitary representation: cyclic $\Rightarrow$ irreducible representation also correct!

Chapter 4: Analysis on locally compact abelian groups.

通常用加法，但这本书仍然用乘法

unimodular, and $f * g(x) = g * f(x) = \int g(y^{-1}x) f(y)\, dy$ (convolution可交换)

$$\| \qquad \int f(y^{-1}x) g(y)\, dy = \int L_y f(x)\, g(y)\, dy$$

Here $L_y f(x) = f(y^{-1}x) = f(xy^{-1})$ 可交换

$R_y f(x) = f(xy) = f(yx)$

4.1: The Dual group.

Recall every irreducible unitary representation on G, must be 1-dimensional (Last lecture)

We may identify $\mathcal{H}_\pi \cong \mathbb{C}$, and $\pi(x) \cdot z = \boxed{\xi(x)} \cdot z \in \mathbb{C}$

$\xi : G \to \mathbb{T} \sim S^1$, continuous group homomorphism called a character.

Denote $\hat{G} = \{\text{all characters of } G\}$ ← later we will equip this with topology, also locally cpt abelian gp.

As $\xi(x) = \langle \xi(x) \cdot 1, 1 \rangle$, by prop 3.15 last lecture $\langle \pi(x)u, u \rangle \in P$, $\xi(x)$ are functions of positive type. 后面用得到

For the reason of symmetry, denote $\xi(x) \overset{\text{def}}{=} \langle x, \xi \rangle$

$\hat{G}$ 也为 locally cpt abelian gp. $\hat{\hat{G}}$ 也可讨论！

Then let $\hat{f}(\xi) = \int_G \langle x, \xi \rangle f(x)\, dx$ ($f \in L^1$) (Recall that $\pi(f)u = \int \pi(x) u f(x) dx \in \mathcal{H}_\pi$, using Riesz representation)

$\pi(f) = \int \pi(x) f(x)\, dx \in \mathcal{B}(\mathcal{H}_\pi)$.

Properties: (special case of thm 3.9, $*$-representation)

① $\widehat{f * g} = \hat{f} \cdot \hat{g}$ ② $\widehat{f^*} = \hat{f}^*$

③ $\xi(x) \cdot \hat{f} = \widehat{L_x f}$

proof: ① $\widehat{f * g}(\xi) = \int_G \int \langle x, \xi \rangle f(y^{-1}x) g(y)\, dx\, dy \overset{\text{unimodular}}{=} \iint \underbrace{\langle yx, \xi \rangle}_{= \langle y, \xi \rangle \langle x, \xi \rangle} f(x) g(y)\, dx\, dy = \hat{f}(\xi) \cdot \hat{g}(\xi)$

being gp homomorphism

② : $\xi(f^*) = \int \langle x,\xi\rangle \overline{f(x^{-1})}\,dx = \overline{\int \underset{\langle x,\xi\rangle}{\overset{\|}{\overline{\langle x^{-1},\xi\rangle}}} f(x)\,dx} = \overline{\xi(f)} = \xi(f)^*$

, As in $\boxed{\mathbb{T}}$, $\alpha^{-1} = \bar{\alpha}$, $\underset{1}{\langle 1,\xi\rangle} = \langle x,\xi\rangle \cdot \langle x^{-1},\xi\rangle$

then $\langle x,\xi\rangle^{-1} = \langle x^{-1},\xi\rangle = \overline{\langle x,\xi\rangle}$

③ : $\xi(x)\cdot\xi(f) = \int \underbrace{\langle x,\xi\rangle\langle y,\xi\rangle}_{\text{gp homo} = \langle xy,\xi\rangle} f(y)\,dy = \int \langle y,\xi\rangle f(x^{-1}y)\,dy = \xi(L_x f)$.

Hence $\xi(f) = \int_G \langle x,\xi\rangle f(x)\,dx$ defines a non-zero multiplicative functional on $L^1(G)$.

may take $f = \overline{\langle x,\xi\rangle}|_{\text{compact}}$.

Now every non-zero multiplicative functional $\hat{\Phi}$ on $L^1(G)$ can be written in the form of the above 找到 character.

also linear.

is given by $\boxed{\text{integration against a character}}$

(analogue to Thm 3.11)

proof: $\forall\, \Phi \in L^1(G)^* \cong L^\infty(G)$, then $\exists$ corresponding $\phi \in L^\infty(G)$. $\Phi(f) = \int \phi\cdot f$

then $\forall f, g \in L^1(G)$

$\int \Phi(f)\,\phi(x) g(x)\,dx = \Phi(f)\Phi(g) = \Phi(f*g)$

$\Downarrow$ Abelian

$= \iint \phi(x)\, f(y) g(y^{-1}x)\,dx\,dy$

$= \iint \phi(yx)\, f(y) g(x)\,dx\,dy = \int (\underbrace{\int \phi(yx) f(y)\,dy}_{\int \phi(y) f(x^{-1}y)\,dy = \Phi(L_x f)})\, g(x)\,dx$

$= \int \Phi(L_x f)\, g(x)\,dx$

Overall $\Phi(f)\phi(x) = \Phi(L_x f) \Rightarrow \boxed{\phi(x) = \frac{\Phi(L_x f)}{\Phi(f)}}$, $\forall f$ s.t. $\Phi(f) \neq 0$

- ϕ being continuous, $\|L_x f - f\|_1 \to 0$, as $x \to 0$
- ϕ being homomorphism. $\phi(xy)\Phi(f) = \Phi(L_{xy} f) = \Phi(L_x L_y f)$

$= \phi(x)\Phi(L_y f) = \phi(x)\phi(y)\Phi(f)$

In particular. $\phi(x^n) = \phi(x)^n$, $\forall n \in \mathbb{Z}$, $\phi \in L^\infty \Rightarrow |\phi| = 1$, or 0

If $\phi(x) = 0$ for some x. $\phi(e) = \phi(x)\cdot\phi(x^{-1}) = 0 \Rightarrow \phi(y) = \phi(1)\cdot\phi(y) = 0$, $\forall y$

$\Rightarrow \Phi = 0$, contradiction!

$\Rightarrow \phi$ is a continuous homomorphism from G to $\mathbb{T}$.

$\boxed{///}$

In short, $\hat{G}$ can be identified with $\{$non-zero mulitiplicative bounded linear functionals on $L^1(G)\} \subset L^1(G)^* \cong L^\infty(G)$

$\xi(f) = \int \langle x,\xi\rangle f(x)\,dx$

$\Downarrow$

Inherits weak-* topology.

From the argument above.

$\hat{G} \cup \{0\} = \{$all multiplicative functionals on $L^1(G)\}$, closed in $L^\infty(G)$. w.r.t weak-* topology.

去掉0以后,仅需 W*Hausdorff (HW)

$\Rightarrow$ W*-compact $\Rightarrow \hat{G}$ is W^* locally cpt. and contained in the $\boxed{\text{unit ball of } L^\infty}$ weak-cpt (Banach-Alaoglu)

Group structure on $\widehat{G}$, $\langle x, \xi_1 \cdot \xi_2 \rangle \overset{\text{def}}{=} \xi_1(x) \cdot \xi_2(x)$, $e = 1_G$ constant function.

$\langle x, \xi_2 \cdot \xi_1 \rangle = \xi_2(x) \cdot \xi_1(x)$

multiplication and inversion are continuous by dominated convergence thm.

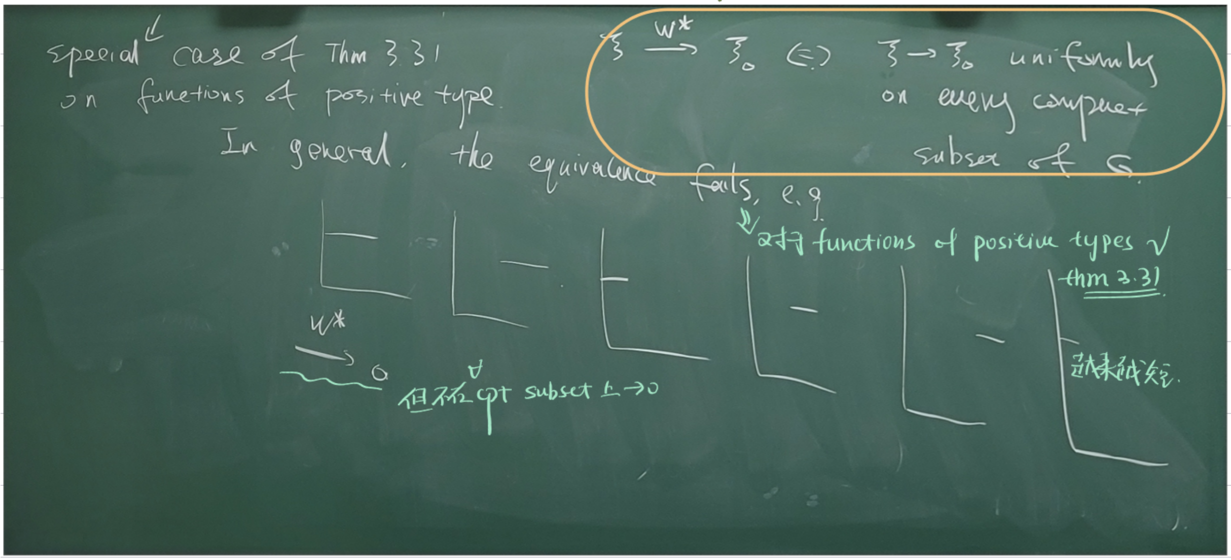
$\Downarrow$

Overall $\widehat{G}$ is a locally compact Abelian group.

More about topology on $\widehat{G}$: the weak-* topology coincides with the compact convergence topology.

$\xi \xrightarrow{w^*} \xi_0 \iff \xi \to \xi_0$ uniformly on every cpt subset of G

In general, the equivalence fails.



$\Leftarrow$

Proof of (11.1): If $\xi \to \xi_0$ uniformly on every compact subset, then $\forall f \in L^1$, $\exists g \in C_c(G)$, $\|f - g\|_{L^1} < \varepsilon$

$$\Rightarrow \left| \int \langle x, \xi - \xi_0 \rangle f(x)\,dx \right| \le \Big| \int_{\underset{\text{cpt}}{\operatorname{supp} g}} \underbrace{\langle x, \xi - \xi_0 \rangle}_{\to 0 \text{ uniformly}} g(x)\,dx \Big| + 2\|f - g\|_{L^1} \to 0$$

$\Rightarrow$, conversely, $\xi \xrightarrow{w^*} \xi_0$, it suffices to show that $\exists$ neighborhood U of $e \in G$.

s.t. $\xi \to \xi_0$ uniformly on U.

Let V be a compact neighborhood of e that will be clarified later. denote $f(x) = \frac{1}{|V|} \chi_V(x)$

then $|V|$ measure $< \infty$

$$|\xi(x) - \xi_0(x)| \le \underbrace{|\xi(x) - f * \xi(x)|}_{\mathrm{I}} + \underbrace{|f * (\xi - \xi_0)(x)|}_{\mathrm{II}} + \underbrace{|\xi_0(x) - f * \xi_0(x)|}_{\boxed{\mathrm{III}}}$$

: ξ_0固定 III最简单

$$\mathrm{III} := \Big| \frac{1}{|V|} \int_V \xi_0(x) - \underbrace{\xi_0(y^{-1}x)}_{\xi_0(y)\xi_0(x)}\,dy \Big| \le \underbrace{\frac{1}{|V|} \int_V |1 - \xi_0(y)|\,dy}_{与x无关} \le \sup_{y \in V} |1 - \xi_0(y)| < \varepsilon$$

When V is "small" (independent of x and ξ)

$\Downarrow$

Now fix V. e.g. V cpt neighborhood $\cap \{y : |1 - \xi_0(y)| < \varepsilon\}$

$$\text{now } \mathrm{I} \le \Big| \frac{1}{|V|} \int_V 1 - \xi(y)\,dy \Big| \le \underbrace{\frac{1}{|V|} \int_V |1 - \xi_0(y)|\,dy}_{\le \varepsilon, \text{ by III}} + \underbrace{\frac{1}{|V|} \Big| \int_V \xi(y) - \xi_0(y)\,dy \Big|}_{\to 0, \text{ as } \xi \xrightarrow{w^*} \xi_0, \text{ independent in } x}$$

It remains to consider II

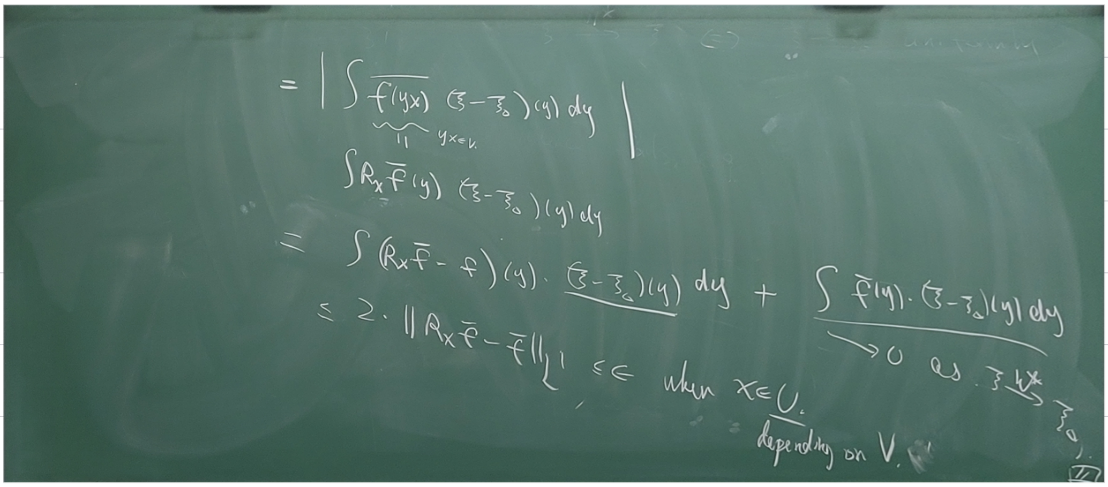
$\nearrow \int f(yx)\overline{(\xi-\xi_0)(y)}\,dy$ $\qquad\qquad$ $\nearrow$ consider conjugate on entire integral.

$$I := \left|\int f(y^{-1}x)\,(\xi-\xi_0)(y)\,dy\right| = \Big|\underbrace{\int \overline{f(yx)}\,(\xi-\xi_0)(y)\,dy}_{\int R_x\bar f(y)\,(\xi-\xi_0)(y)\,dy}\Big|$$

$$= \int (R_x\bar f - \bar f)(y)\,(\xi-\xi_0)(y)\,dy + \underbrace{\int \bar f(y)\,(\xi-\xi_0)(y)\,dy}_{\to 0,\ \text{as } \xi \xrightarrow{w^*} \xi_0}$$

$$\le 2\cdot\|R_x\bar f - \bar f\|_{L^1} < \varepsilon, \quad \text{when } x\in U, \text{ depending on } V.$$

$\boxed{\text{III}}$



Lecture 19: cont. Characters

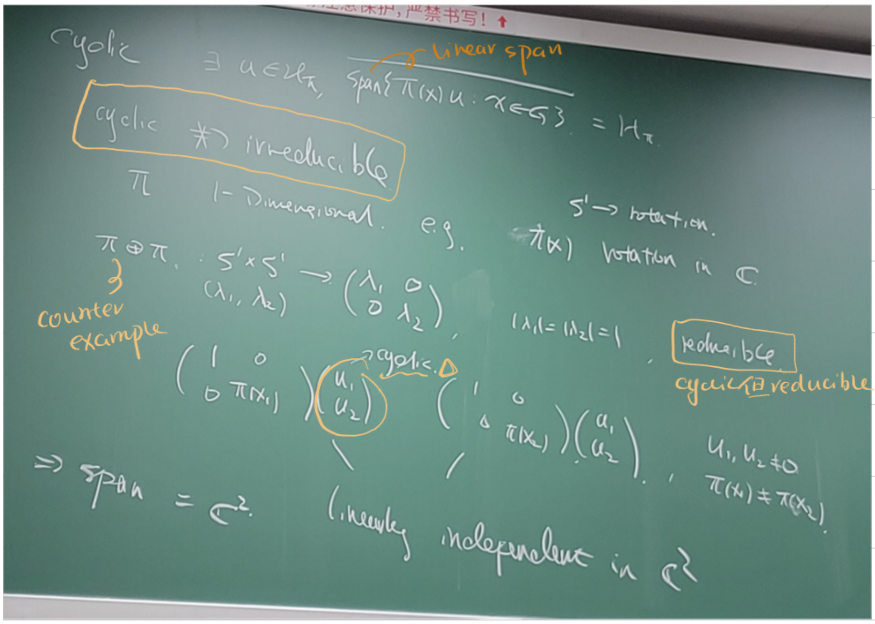
Lecture 19-2023 年 5 月 9 日降温 ☼ 22°C-26°C

主要内容： G 和 $\hat{G}$ 之间的关系（尤其 G 为 compact 的时候）

其他信息：

Summer school: 见录音 (大概在7月底 3~5天)

之前的刊误 也有问题



last lecture: $\widehat{G}$: dual group = {characters} $\xi_1 \cdot \xi_2(x) = \xi_1(x) \cdot \xi_2(x) = \langle x, \xi_1 \rangle \cdot \langle x, \xi_2 \rangle$

$\widehat{G} \cup \{0\}$ is compact in $L^\infty \cong (L^1)^*$

Prop 4.5: If G is compact then $\widehat{G}$ is discrete.

需要 prop 4.4 If G is discrete. then $\widehat{G}$ is compact.

Prop 4.4: If G is compact, then $\widehat{G}$ is an orthonormal set in $L^2(G)$

$\widehat{G} \subset L^\infty(G) \subset L^2(G)$

Proof of prop 4.4: Since $\int \xi \cdot \bar{\eta} = \int \langle x, \xi\eta^{-1} \rangle dx$

$x_0 \cdot x_0^{-1} \cdot x$, then by group homomorphism

$$= \langle x_0, \xi\eta^{-1} \rangle \int \langle x_0^{-1}x, \xi\eta^{-1} \rangle dx \quad \text{by left-Haar measure.}$$

$$= \langle x_0, \xi\eta^{-1} \rangle \int \langle x, \xi\eta^{-1} \rangle dx = \langle x_0, \xi\eta^{-1} \rangle \int \xi \cdot \bar{\eta}$$

$$\Rightarrow \int \xi \cdot \bar{\eta} = \begin{cases} 1, & \xi = \eta \\ 0, & \text{otherwise} \end{cases}$$ ▥

Now we may see the proof of prop 4.5

Proof of prop 4.5: If G is compact, then $1_G \perp \xi$, $\forall \xi \in \widehat{G} \setminus \{1_G\}$, then

$\{1_G\} = \overline{\widehat{G} \setminus \{1_G\}}^c$, then $\{1_G\}$ is open

⇓

单点集 $\{e\}$ is open in $\widehat{G}$ ⇒ $\widehat{G}$ is discrete.

Now, suppose G is discrete, then $L^1(G)$ has a [illegible] $\delta(x) = \begin{cases} 1, & x = e \\ 0, & \text{otherwise} \end{cases}$

由 G being discrete, δ为一个function!

then there is no way for $\xi \xrightarrow{w^*} 0$, $\xi \in \widehat{G}$ ⟺ $\langle \xi, \delta \rangle \to 0$) but $\langle \xi, \delta \rangle = \delta(\xi)$ = 1

$\Rightarrow \{0\}$ is an isolation point in $\widehat{G} \cup \{0\}$, and recall that $\widehat{G} \cup \{0\}$ is compact in L^∞

$\Rightarrow \widehat{G}$ is compact.

Remark ①: $\int f(x)\, \overline{\langle x, \xi \rangle}\, dx \in C_0(\widehat{G})$, $\hat{\delta} = 1 \Rightarrow \widehat{G}$ is compact. (Folland, Rudin 的证明, 以上为另一种证明)

Remark ②: The second half of this proof actually says $\underbrace{\delta}_{\text{multiplicative identity}} \in L^1(G) \Rightarrow \widehat{G}$ is compact.

Together with $\boxed{\text{Pontrjagin duality } \widehat{\widehat{G}} \cong G}$, we can conclude $\delta \in L^1(G)$ iff G is discrete

proof: "$\Leftarrow$" the 2nd half

"$\Rightarrow$" $\delta \in L^1(G) \xRightarrow{\text{2nd-half}} \widehat{G}$ is compact $\xRightarrow{\text{1st-part}} \widehat{\widehat{G}}$ is discrete.

$\widehat{\widehat{G}} \overset{\text{Pontrjagin}}{\cong} G$

E.g. (Thm 4.6)

a: $\widehat{\mathbb{R}} \cong \mathbb{R}$, $\langle x, \xi \rangle = e^{2\pi i \xi \cdot x}$

b: $\widehat{\mathbb{T}} \cong \mathbb{Z}$, $\langle \alpha, n \rangle = \alpha^n$

c: $\widehat{\mathbb{Z}} \cong \mathbb{T}$, $\langle n, \alpha \rangle = \alpha^n$

d: $\widehat{\mathbb{Z}_K} \cong \mathbb{Z}_K$, $\langle m, n \rangle = e^{2\pi i\, m \cdot n / K}$

proof: (a): $\forall \phi \in \widehat{\mathbb{R}}$, $\phi(0) = 1$, so $\exists a > 0$ s.t. $\int_0^a \phi \overset{\text{def}}{=} A \neq 0$

then $A \cdot \phi(x) = \int_0^a \phi(t)\phi(x)\, dt = \int_0^a \phi(t+x)\, dt$

$$\underbrace{= \int_x^{a+x} \phi(t)\, dt}_{\Downarrow}$$

take derivative $A\phi'(x) = \underbrace{\phi(a+x)}_{\phi(a)\cdot\phi(x)} - \phi(x) = (\phi(a) - 1)\phi(x) \Rightarrow \phi$ of the exponential form 微分方程

Since $|\phi| = 1$, then $\phi = e^{2\pi i \xi \cdot x}$ for some $\xi \in \mathbb{R}$.

Conversely, every $\xi \in \mathbb{R}$, $e^{2\pi i x \cdot \xi}$ is a character on $\mathbb{R}$.

(b). $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, note that

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{e^{2\pi i \xi x}} & \\ \downarrow & \circlearrowright & \nearrow \text{提升} \\ \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathbb{T} \end{array}$$

$\Rightarrow \xi \in \mathbb{Z}$ (需要周期性)

For any characters $\Rightarrow e^{2\pi i n \cdot x}$

Conversely, every $e^{2\pi i n \cdot x}$, $n \in \mathbb{Z}$ is a character.

(c) $\widehat{\mathbb{Z}} \cong \mathbb{T}$. $\forall \phi \in \widehat{\mathbb{Z}}$, take $\underbrace{\phi(1) \overset{\text{def}}{=} \alpha \text{ (}\alpha \in \mathbb{T}\text{ 的 character)}, \text{ then } \phi(-1) = \alpha^{-1}}_{\phi(n) = \alpha^n}$

(d) $\mathbb{Z}_K \cong \mathbb{Z}/K\mathbb{Z}$.

Prop 4.7: $\widehat{(G_1\times\cdots\times G_n)} \cong \widehat{G_1}\times\widehat{G_2}\times\cdots\times\widehat{G_n}$, $\{G_i\}$ are locally compact abelian group

Proof: $\xi_i\in\widehat{G_i}$, then $\langle (x_1,\dots,x_n),(\xi_1,\dots,\xi_n)\rangle \overset{\text{def}}{=} \prod_{i=1}^{n}\langle x_i,\xi_i\rangle$ defines a character

Conversely, every character χ on $G_1\times\cdots\times G_n$ can be written as

$$\chi(x_1,\dots,x_n)=\prod_{i=1}^{n}\chi_i(x_i) \text{ where } \chi_i(x)=\chi(e,e,\dots,\underset{\uparrow \text{ i-th position}}{x},e,\dots) \text{ is a character on } G_i$$

$\chi(x,y)=\chi(e,y)\cdot\chi(x,e)$ ▥

Application: $\widehat{\mathbb{R}^n}\cong\mathbb{R}^n$, $\langle x,\xi\rangle=e^{2\pi i x\cdot\xi}$

$\widehat{\mathbb{T}^n}\cong\mathbb{Z}^n$, $\langle z,n\rangle=z^n$

$\widehat{\mathbb{Z}^n}\cong\mathbb{T}^n$

therefore $\widehat{G}\cong G$, for any finite abelian group.

Prop 4.9: G_α compact, then $\widehat{\prod_{\alpha\in A}G_\alpha}=\bigoplus_{\alpha\in A}\widehat{G_\alpha}$

$\| \; \{\prod_{\alpha\in A}\xi_\alpha,\ \xi_\alpha\in\widehat{G_\alpha}$, and $\xi_\alpha=e_\alpha$ for all but finitely many $\alpha\}$

characters

proof: $\forall \xi\in\widehat{\prod G_\alpha}$, let ξ_α be its restriction on G_α, we shall prove that

$\xi_\alpha=e_\alpha$ for all but finitely many α.

Consider $\{x\in\prod G_\alpha : |\langle x,\xi\rangle-1|<1\}$ a neighborhood of e

contains $\prod V_\alpha$ (e.g. topology basis), V_α neighborhood of e_α. $V_\alpha=G_\alpha$ for all but finitely many α

$\Rightarrow$ If $V_\alpha=G_\alpha$,

$|\xi_\alpha(G_\alpha)-1|<1 \quad\Rightarrow\quad \xi_\alpha=e_\alpha$ as $\xi_\alpha(G_\alpha)$ is a subgroup of $\mathbb{T}$

$\| \; \xi(\prod x_\beta)$, $x_\beta=\begin{cases}G_\alpha, & \alpha=\beta\\ e_\beta, & \text{otherwise.}\end{cases}$ ▥

Example: $(\mathbb{Z}/_2)^{\omega}$ → countable

Explanation 1: each character ξ on $\mathbb{Z}/_2$ is $\xi(x)=1$, or $(-1)^x$, $x=0,1$

every ξ on $(\mathbb{Z}/_2)^{\omega}$, $\xi=\prod_n\xi_n$. ξ_n is trivial for all but finitely many n.

Explanation 2: via dyadic decomposition

0 —— 1

$[0,\frac12)$ $\quad [\frac12,1)$

$[0,\frac14)$ $[\frac14,\frac12)$ $[\frac12,\frac34)$ $[\frac34,1)$

base-2 decimal expansion

Lebesgue measure $\leftrightarrow$ Haar measure

$\underset{[0,1)}{\sum a_j 2^{-j}} \mapsto (a_j)\in(\mathbb{Z}/_2)^{\omega}$

↓ not a group homomorphism.

What about characters?

non-trivial ξ_n becomes n-th Rademacher function r_n that equals

$$1, -1, 1, -1, \cdots \text{ on } [j2^{-n}, (j+1)2^{-n}), \ j = 0, 1, \cdots, 2^n - 1$$

and $(\widehat{\mathbb{Z}_2})^{\omega} \rightsquigarrow \{\underset{\text{product}}{\prod_{\text{finite}}} r_n\} \overset{\text{def}}{=} \{w_n\}_{n=0}^{\infty}$

$$w_0 = 1, \ w_n = r_1^{b_1} r_2^{b_2} \cdots r_k^{b_k}, \ n = \sum_{j=1}^{k} b_j \cdot 2^{j-1}$$

$\Rightarrow \{w_n\}_{n=0}^{\infty}$ is an orthonormal family in $L^2([0,1))$. By plancherel, it is an orthonormal basis.

Walsh functions

平移加法测度 measure 一致

next section

Example $\mathbb{Q}_p$, $r = p^m \cdot \frac{a}{b}$, $(a, b) = 1$, $p \nmid a$, $|r|_p = p^{-m}$

$$|r_1 + r_2| \le \max\{|r_1|, |r_2|\}, \ |r_1 \cdot r_2| = |r_1| \cdot |r_2|, \ \mathbb{Q}_p = \overline{\mathbb{Q}}$$

and $\mathbb{Q}_p = \{\underbrace{\sum_{j \ge m}^{\infty} c_j p^j}_{\text{series form}}, m \in \mathbb{Z}, c_j = 0, 1, \cdots, p-1\}$

Also $B(\underset{\text{radius}}{1}, \underset{\text{center}}{0}) \overset{\text{def}}{=} \underset{\text{integer ring}}{\mathbb{Z}_p} = \{\sum_{j \ge 0}^{\infty} c_j p^j, c_j = 0, 1, \cdots, p-1\}$

open subgroup

$$B(p^{-k}, 0) = \{\sum_{j \ge k}^{\infty} c_j p^j, \ c_j = 0, 1, \cdots, p-1\}$$

We first find a character ξ_1 by $\langle x, \xi_1 \rangle = e^{2\pi i x}$

$$\text{if } x = \sum c_j p^j, \ e^{2\pi i \sum c_j p^j} = e^{2\pi i \underbrace{\sum_{j \le -1}}_{\text{finite sum}} c_j p^j}$$

clearly $\langle x + y, \xi_1 \rangle = \langle x, \xi_1 \rangle + \langle y, \xi_1 \rangle$

ξ_1 is a constant on every coset of $\underbrace{\mathbb{Z}_p}_{\text{open}}$. $\Rightarrow$ then ξ_1 is continuous.

For $y \in \mathbb{Q}_p$, let $\langle x, \xi_y \rangle = \langle xy, \xi_1 \rangle$, also a character

We shall prove $\underset{\mathbb{Q}_p}{y} \longmapsto \underset{\widehat{\mathbb{Q}_p}}{\xi_y}$ is an isomorphism between topology group $\mathbb{Q}_p$, $\widehat{\mathbb{Q}_p}$

In particular $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$.

Next lecture.

Lecture 20: $\widehat{\mathbb{Q}}_p \sim \mathbb{Q}_p$, Fourier Transform

Lecture 20-2023 年 5 月 11 日凉 22℃-26℃

主要内容：证明 $\widehat{\mathbb{Q}}_p$ 与 $\mathbb{Q}_p$ 同构，Fourier transform 也是基于 characters 的

其他信息：

Recall last time $\widehat{\underbrace{\prod_\alpha}_{\text{product}} G_\alpha} = \underbrace{\bigoplus_\alpha}_{\text{direct sum}} \widehat{G_\alpha}$

不一样 → 若 G_α 为compact, 则 $\prod_\alpha G_\alpha$ cpt, 且 $\widehat{\prod_\alpha G_\alpha}$ 为discrete, 因此右端需要为 direct-sum, 不然 不能保证单点集为开集 (discrete)

cont. $\mathbb{Q}_p = \{\sum_{j\geq m} c_j p^j,\ m\in\mathbb{Z},\ c_j = 0,1,\cdots,p-1\}$

let $\xi_1(x) \overset{def}{=} e^{2\pi i \sum_{j=N}^{-1} c_j p^j}$, if $x = \sum_{j\geq N} c_j p^j$, $\xi_y(x) \overset{def}{=} e^{2\pi i y x}$

$\qquad \| $

$\quad e^{2\pi i x}$ $\qquad\qquad\qquad\qquad \langle x, \xi_y\rangle = \langle xy, \xi_1\rangle$

$\boxed{\text{Goal:}}$ $y \mapsto \xi_y$ is isomorphism

$\mathbb{Q}_p \cong \widehat{\mathbb{Q}_p}$

Lemma 4.10: If $\xi \in \widehat{\mathbb{Q}_p}$, $\exists k\in\mathbb{Z}$, s.t. $\xi = 1$ on $B(p^{-k}, 0)$

proof: $\exists k$ s.t. $|\xi(x) - 1| < 1$, $\forall x \in \underbrace{B(p^{-k}, 0)}_{\text{open subgroup}}$

$\Rightarrow \xi(B(p^{-k},0))$ is a subgroup $\Rightarrow \xi = 1$ on $B(p^{-k},0)$ ▨

Remark: ① $\forall \xi$, $\exists j_0$ s.t. $\xi(p^j) = 1$, $\forall j \geq j_0$, $\xi(p^{j_0-1}) \neq 1$

② $\Rightarrow \xi$ is a constant on every ball of radius p^{-k}, so ξ is determined by its value on p^j, $j\in\mathbb{Z}$

$\xi(\sum_{j\geq m} c_j p^j) = \prod_{j=m}^{k-1} \xi(p^j)^{c_j}$

We start from $j_0 = 0$.

Lemma 4.11: Suppose $\underset{W_0}{\langle 1, \xi\rangle} = 1$, $\underset{W_1}{\langle p^{-1}, \xi\rangle} \neq 1$, then

$$\langle p^{-k}, \xi\rangle = e^{2\pi i \sum_{j=1}^{k} c_{k-j} p^{-j}} \quad \forall 1,2,3,\cdots, \text{ for some } c_j \in \{0,1,\cdots,p-1\}\ j=1,2,\cdots,\ c_0\neq 0$$

proof: Denote $W_k = \langle p^{-k}, \xi\rangle$, then

$$W_k = \langle p^{-k}, \xi\rangle = \langle p\cdot p^{-k-1}, \xi\rangle = \langle p^{-k-1}, \xi\rangle^p = W_{k+1}^p$$

$1 = W_0 = W_1^p \Rightarrow W_1 = e^{2\pi i c_0/p}$, $c_0 \neq 0$

$W_1 = W_2^p \Rightarrow W_2 = e^{2\pi i (\frac{c_0}{p^2} + \frac{c_1}{p})}$ $(c_0 + c_1 p)\cdot p^{-2}$, $c_1 \in \{0,1,\cdots,p-1\}$

$W_2 = W_3^p \Rightarrow W_3 = e^{2\pi i (\frac{c_0}{p^3} + \frac{c_1}{p^2} + \frac{c_2}{p})}$ $\quad c_2 \in \{0,1,\cdots,p-1\}$

$(c_0 + c_1 p + c_2)\cdot p^{-3}$

$\cdots$ done by induction! ▨

Now,

Lemma 4.12: $\xi \in \widehat{\mathbb{Q}_p}$, $\langle 1, \xi\rangle = 1$, $\langle p^{-1}, \xi\rangle \neq 1$, then $\exists \underset{|y|=1}{y \in \mathbb{Q}_p}$ s.t. $\xi = \xi_y$.

proof: Take $y = c_0 + c_1 p + c_2 p^2 + \cdots$, $|y| = 1$, and

$$\langle p^{-k}, \xi\rangle = e^{2\pi i (c_0 + c_1 p + \cdots + c_k p^k)\cdot p^{-k}} = \langle y\cdot p^{-k}, \xi_1\rangle = \langle p^{-k}, \xi_y\rangle \Rightarrow \xi = \xi_y.$$ ▨

Thm 4.13: $y \mapsto \xi_y$ is an isomorphism between $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}_p}$

proof: group homomorphism ✓

Injective ✓ $\langle x, \xi_y \rangle = e^{2\pi i x \cdot y}$ ✓

Now we show that it is surjective: $\forall \xi \in \widehat{\mathbb{Q}_p}$, $\exists$ smallest integer j s.t.

$\langle p^j, \xi \rangle = 1$, then consider η, s.t. $\underbrace{\langle x, \eta \rangle = \langle p^j x, \xi \rangle}_{\text{character}}$

By previous lemma, $\eta = \xi_y$, for some $y \in \mathbb{Q}_p$, $|y| = 1$

$\Rightarrow \langle x, \xi \rangle = \langle p^j p^{-j} x, \xi \rangle = \langle p^{-j} x, \eta \rangle = \langle p^{-j} x, \xi_y \rangle = \langle x, \xi_{p^{-j} y} \rangle$

$\Rightarrow \xi = \xi_{p^{-j} y}$

Hence $y \mapsto \xi_y$ is a group isomorphism. It remains to show that it is homeomorphism. (topology)

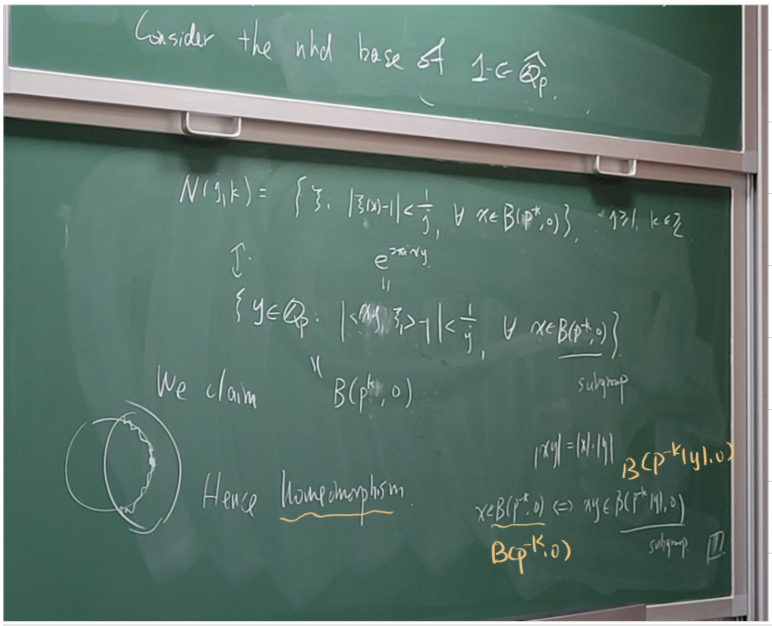
Consider the neighborhood base of $1 \in \widehat{\mathbb{Q}_p}$

weak* topo

$$N(j,k) = \{\xi : |\xi(x) - 1| < \tfrac{1}{j},\ \forall x \in B(p^{-k}, 0)\},\ j \geq 1,\ k \in \mathbb{Z}$$

topology basis对应 $\Updownarrow$

$$\{y \in \mathbb{Q}_p : |\underbrace{\langle xy, \xi_1 \rangle}_{= e^{2\pi i x y}} - 1| < \tfrac{1}{j},\ \forall x \in \underbrace{B(p^{-k}, 0)}_{\text{subgroup}}\} = B(p^k, 0)$$

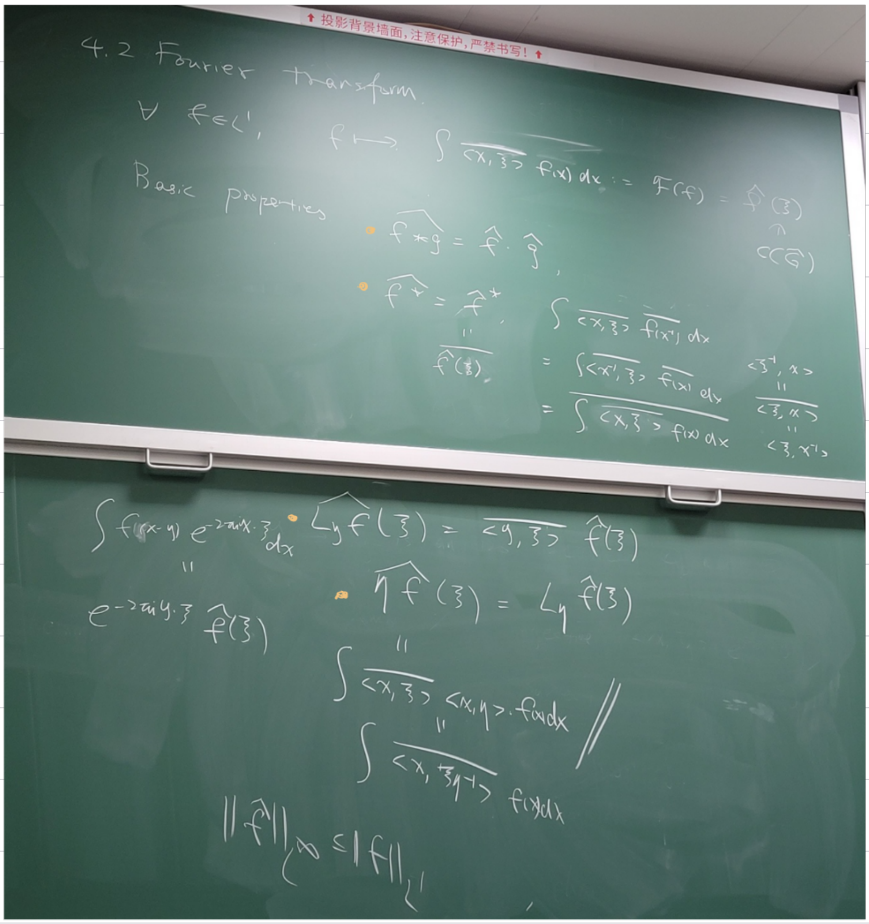


Section 4.2: Fourier Transform

$\forall f \in L^1$, $f \mapsto \int \overline{\langle x, \xi \rangle} f(x)\,dx \overset{\text{def}}{=} \mathcal{F}(f) = \hat{f}(\xi) \in C(\hat{G})$

Basic properties:

- $\widehat{f * g} = \hat{f} \cdot \hat{g}$
- $\widehat{f^*} = \hat{f}^* = \overline{\hat{f}(\xi)}$
- $\widehat{L_y f}(\xi) = \overline{\langle y, \xi \rangle} \hat{f}(\xi)$



Prop 4.18: $\hat f\in C_0(\hat G)$, and $\mathcal{F}(L^1(G))$ is a dense subset of $C_0(\hat G)$ (by stone-weierstrass)

proof: $\int\overline{\langle x,\xi\rangle}f(x)dx$ is continuous in $\xi\in L^\infty$ under weak*-topology

continuous in $\hat G\cup\{0\}$

Recall that $\hat G\cup\{0\}$ is compact in L^∞ under w-* topology

then $\int\cdots f(x)dx\to 0$, $\Rightarrow\hat f(\xi)\in C_0(\hat G)$ ▯

Recall in $\mathbb{R}^n$ (之前的证法), $\forall f\in L^1$, $\exists g\in C_0^\infty$ s.t. $\|f-g\|_{L^1}<\varepsilon$

$$\hat f=\underset{C_0}{\hat g}+\underbrace{\int(f-g)e^{2\pi i}}_{<\varepsilon}$$

More generally, one can define the Fourier transform on $M(G)$, finite complex Radon measure μ.

$$\hat\mu(\xi)=\int\overline{\langle x,\xi\rangle}\,d\mu(x)\in C(\hat G)$$

then

$$\widehat{\mu*\nu}=\hat\mu\cdot\hat\nu,\quad \int\phi\,d(\mu*\nu)=\iint\phi(xy)\,d\mu(x)\,d\nu(y)$$

(ϕ bounded continuous)

$$\|\hat\mu\|_{L^\infty}\le\|\mu\|_{M(G)}$$

Similarly, $\forall\mu\in M(\hat G)$, one can define $\phi_\mu(x)=\int\langle x,\xi\rangle\,d\mu(\xi)\in C(G)$

Prop 4.18: the map $\mu\to\phi_\mu$ is injective from $M(\hat G)$ to $C(G)$

proof: If $\phi_\mu=0$, $0=\int f\overline{\phi_\mu}=\iint\underbrace{f(x)}_{\in L^1(G)}\overline{\langle x,\xi\rangle}\,d\mu(\xi)\,dx\overset{FS}{=}\int\underbrace{\hat f(\xi^{-1})}_{\text{dense in }C_0(\hat G)}d\mu(\xi)$, for $\forall f\in L^1(G)$

then $\mu=0$ ▯

If $\mu \in M(\hat{G})$ is positive, then ϕ_μ is a function of positive type

i.e. $\int (f * f^*(x)) \phi_\mu(x)\, dx \geq 0$

$= \iint f * f^*(x) \cdot \overline{\langle x, \xi^{-1} \rangle}\, dx\, d\mu(\xi) = \int |\hat{f}(\xi^{-1})|^2 d\mu(\xi) \geq 0$

Thm 4.19 (Bochner's thm)

If $\phi \in P(G)$, $\exists!\, \mu \in M(\hat{G})$ positive s.t. $\phi = \phi_\mu$.

- In the proof we need $\|\underbrace{f * \cdots * f}_{n}\|_{L^1}^{\frac{1}{n}} \to \|\hat{f}\|_{L^\infty}$, $\forall f \in L^1$

  In fact, it can be extended to $\mu \in M(G)$.

Recall that: $\sigma(x) \overset{\text{def}}{=} \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}$

$R(\lambda) = (\lambda e - x)^{-1}$, analytic in $\lambda \in \mathbb{C} \setminus \sigma(x)$

$\rho(x) \overset{\text{def}}{=} \sup\{|\lambda| : \lambda \in \sigma(x)\} \leq \|x\|$

谱半径

Thm 1.8: $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$

proof: $\lambda^n e - x^n = (\lambda e - x) \cdot \sum_0^{n-1} \lambda^j x^{n-1-j}$

$= \sum_0^{n-1} \lambda^j x^{n-1-j} (\lambda e - x)$

So $\lambda^n e - x^n$ is invertible $\Rightarrow$ $\lambda e - x$ is invertible.

So $\lambda \in \sigma(x) \Rightarrow \lambda^n \in \sigma(x^n)$, then $|\lambda^n| \leq \|x^n\|$

$\Rightarrow \|x^n\|^{\frac{1}{n}} \geq |\lambda|$, $\forall \lambda \in \sigma(x)$.

$\Rightarrow \liminf \|x^n\|^{\frac{1}{n}} \geq \rho(x)$.

Conversely, $\forall$ bounded linear functional ϕ, $\phi \circ R(\lambda)$ is analytic in $\lambda \in \mathbb{C} \setminus \sigma(x)$

In particular, it is analytic in $\lambda > \rho(x)$.

Recall, when $|\lambda| > \|x\|$

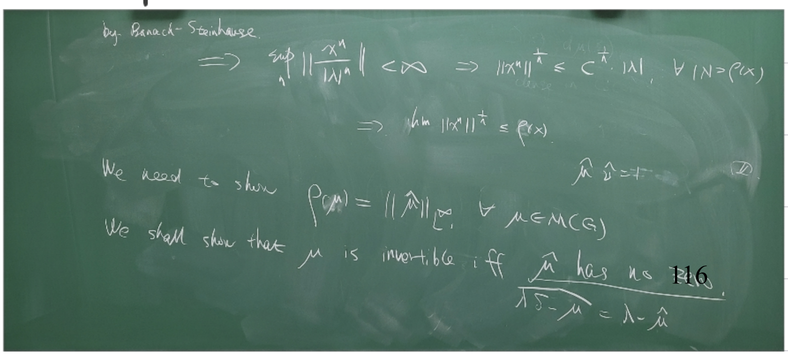
$\lambda e - x = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n \Rightarrow \underbrace{\phi \circ R(\lambda)}_{\text{analytic in } |\lambda| > \rho(x)} = \sum_{n=0}^{\infty} \lambda^{-n-1} \phi(x^n)$, $\forall |\lambda| \geq \|x\|$

$\Downarrow$ by complex analysis

absolutely convergent in $|\lambda| > \rho(x)$

$\Rightarrow |\phi(x^n)| \leq C_\phi \cdot |\lambda|^{n+1}$

Now, by Banach-Steinhaus



Lecture 21: Bochner's theorem, Fourier Inversion

Lecture 21-2023 年 5 月 18 日热 ☼ 26°C-31°C

主要内容：**Bochner's theorem**; Fourier Inversion Theorem I (证明需要 Lemma 4.20, Lemma 4.21)

之后我们才能利用这个 Version 1 得到 Pontrjagin duality

其他信息：本节课因连花清瘟胶囊（4 颗）的胃肠道副作用未能去线下课，本笔记基于线上录屏补充（线上录屏的比例有问题，基本属于不可用的状态）；本次电子笔记首次尝试使用 iPad Pro（发现有几个问题，iPad Pro 11 寸配合 Apple Pencil 并不是太适用 Galaxy Tab 的笔记模板，因为 iPad 写出的字更大，可能需要使用更小的纸张），而且貌似 iPad Pro 导出的 PDF（因为都是矢量字体，对 PDF 浏览器的渲染要求更大，不过印刷出来的效果确实会更好一些）

Last time $\underbrace{\|f*\cdots*f\|}_{n\text{-fold}}{}_{L^1}^{\frac{1}{n}} \to \|\hat f\|_{L^\infty}$, $\forall f\in L^1$

this result is also carried in more abstract Banach algebra (may require being Abelian)

Here we only consider locally compact abelian group

Proving the above result requires spectrum radius theorem than

$L^1(G)$ $\quad \|x^n\|^{\frac{1}{n}} \to \rho(x)$ spectrum radius

We shall prove that $f\in L^1$ $\boxed{\hookrightarrow M(G)}$ then we have identity

Finite complex Radon measure

Now we want to show that $\boxed{\mu \text{ has an inverse in } M(G) \Leftrightarrow \hat\mu \text{ has no zero}}$ what we need to show now.

If so, then $\rho(\mu)=\|\hat\mu\|_{L^\infty}$

考 $\lambda\delta-\mu$ 可逆 对应 dirac-delta

Recall that μ has an inverse ν: $\mu*\nu=\delta$, then $\hat\mu\hat\nu=1$, now support

$\hat\mu$ has a zero, then $\hat\mu\hat\nu\neq 1$, then μ is not invertible! therefore we have shown the $\Rightarrow$ side.

Now it suffices to show "$\Leftarrow$", that is 不易证 if μ is not invertible, then $\hat\mu$ has a zero

这部分在书上是之前 general theory 的一部分!

$\exists$ a characters $\xi\in\hat G$ s.t $\int \underbrace{\langle x,\xi\rangle}_{\hat\xi(x)} d\mu(x)=0$

If μ is not invertible then μ is contained in a maximal ideal J. (by Zorn's lemma) 即包含于一个 ideal, 但这 ideal 不包含 identity (proper ideal) (: 不 invertible)

and the maximal ideal J is non-trivial. Notice that $J\subset\{\text{all non-invertible elements}\}$

Also an ideal.

and a closed subset, proper as it does not contain identity.

As invertible $\to$ open subset

$\Rightarrow \mu\in$ closed maximal ideal J

consider mod

now we consider $\boxed{M(G)/J}$ closed, with quotient norm

Still a Banach algebra, since $\quad \|\tilde f\|=\min_{g\in J}\|f+g\|$

J is maximal. We claim that $M(G)/J$ is one-dimensional (Field)

by Former lecture

therefore it is isomorphic to $\mathbb{C}$, i.e. $M(G)/J\cong\mathbb{C}$

Now we have a natural map $\pi: M(G)/J\to\mathbb{C}$, where $\ker(\pi)=J$

can be lifted

$$\begin{tikzcd} M(G) \arrow[d, "P"] \arrow[r, dashed] & \boxed{\tilde\pi} \\ M(G)/J \arrow[r, "\pi"] & \mathbb{C} \end{tikzcd}$$

$\forall$ multiplicative bounded linear functional

therefore $\exists$ a character $\xi\in\hat G$, s.t $\tilde\pi(f)=\int\langle x,\xi\rangle f(x)dx$, $\forall f\in L^1(G)$

Note that $L^1(G)$ is dense in $M(G)$, we could have $\tilde\pi(\nu)=\int\langle x,\xi\rangle d\nu(x)$ (Extended to $M(G)$)

for $\forall\nu\in M(G)$

As $\ker(\tilde\pi)=J\ni\mu \Rightarrow 0=\tilde\pi(\mu)=\int\langle x,\xi\rangle d\mu(x)=\hat\mu(\xi)$

$\boxed{\hat\mu \text{ has a zero.}}$

Overall, the zero of $\hat{\mu}$ corresponds to those maximal ideals

e.g. Given $\xi \in \hat{G}$, then $\{\mu \in M(G), \hat{\mu}(\xi) = 0\}$ $\Leftrightarrow$ maximal ideal ∎

The above result is for the proof of the Bochner's theorem

Thm 4.19 (Bochner) functions of positive type $\int (f^* * f)\cdot\phi \geq 0 = \int |\hat{f}(\xi)|^2 d\mu(\xi)$ if $\phi = \phi_\mu$ (we have shown the reverse of Bochner's thm)

If $\phi \in P(G)$, $\exists!\ \mu \in M(\hat{G})$, s.t. $\phi = \phi_\mu$ a.e.

$\int \langle x, \xi \rangle d\mu(x)$ 某个正测度的 Fourier transform

proof: Notice that if we have ψ_U is an approximating identity so is $\psi_U^* * \psi_U$ is also an approximating identity.

为了与 positive type 联系

Consider Hermitian form $(f, g)_\phi = \int \phi (g^* * f)$

By Cauchy-Schwarz $|(f, g)_\phi|^2 \leq |(f, f)_\phi| \cdot |(g, g)_\phi|$

$$\left|\int \phi (g^* * f)\right|^2 \leq \left(\int \phi (f^* * f)\right)\left(\int \phi (g^* * g)\right)$$

let $g = \psi_U$, and we may assume $\underbrace{\int \phi (g^* * g)}_{\text{approximating}} \to C$ (in fact $\phi(0)$)

$\Rightarrow$ take limit on both sides of the inequality

$$\left|\int \phi f\right|^2 \leq C \left(\int \phi (f^* * f)\right) \quad \forall f \in L^1$$

$$\Downarrow$$

$$\boxed{\left|\int \phi f\right|} \leq C^{\frac{1}{2}} \left(\int \phi\, \underbrace{(f^* * f)}_{\text{consider this } f}\right)^{\frac{1}{2}}$$

$$\leq C^{\frac{1}{2}+\frac{1}{4}} \left(\int \phi (f^* * f * f^* * f)\right)^{\frac{1}{4}}$$

Denote $h = f^* * f$, then

$$\leq C \cdot \lim_{n\to\infty} \left|\int \phi h^{(n)}\right|^{\frac{1}{2n}} \quad L^\infty \neq 0$$

$$\leq C \cdot \lim_{n\to\infty} \|h^{(n)}\|_{L^1}^{\frac{1}{2n}} = C \cdot \|\hat{h}\|_{L^\infty}^{\frac{1}{2}} \quad \text{by previous result}$$

$$= \boxed{C \cdot \|\hat{f}\|_{L^\infty}}$$

$$\Downarrow$$

Recall $f \mapsto \hat{f} \in C_0(\hat{G})$ is injective (4.18)

$f \in L^1(G)$ and $\{\hat{f} : f \in L^1\}$ is dense in $C_0(\hat{G})$

$\Rightarrow$ ϕ defines a bounded linear functionals on $C_0(\hat{G})$ (唯一 extend)

$\Downarrow$

complex finite Radon measure

$\Downarrow$

$\exists\ \mu \in M(\hat{G})$ s.t. $\int \phi f = \int \hat{f}(\xi) d\mu$

$$= \int\int \overline{\langle x, \xi \rangle} f(x) dx\, d\mu(\xi)$$

$$= \int f(x) \hat{\mu}(x)\, dx$$

$\Rightarrow \phi = \hat{\mu}$ ∎

Now we will spend a lot of time proving the Fourier inversion.

We first prove some partial results in Fourier Inversion $\Longrightarrow$ $\boxed{\text{Pontrjagin Duality}}$ $\Rightarrow$ Full Form of Fourier Inversion

The most familiar form of Fourier inversion ($f\in L^1$, $\hat f\in L^1 \Rightarrow \hat{\hat f}(x) = f(x^{-1})$, a.e.) 即

空间 matters

We shall prove another version first:

Denote $B(G) = \{\phi_\mu,\ \mu\in M(\hat G)\}$ = linear span of $P(G)$ (Bochner; $\phi_\mu = \int\langle x,\xi\rangle d\mu(\xi)$)

$B^1(G) = B(G)\cap L^1(G)$

As $\mu\in M(\hat G)\to\phi_\mu\in B(G)$ is a Bijection, Denote its inverse by $\phi\mapsto\mu_\phi$

i.e. $\phi(x) = \int\langle x,\xi\rangle d\mu_\phi(\xi)$ (∃ such μ_ϕ)

Thm 4.22 (Fourier Inversion theorem 1)

If $f\in B^1$, then $\hat f\in L^1(\hat G)$, and $f(x) = \int\langle x,\xi\rangle\hat f(\xi)d\xi$ if the Haar measure is suitably normalized.

μ 就是 $\hat f$

$d\mu_f(\xi) = \hat f(\xi)d\xi$ (a sense of Fourier inversion)

Lemma 4.20: If $K\subset\hat G$ is compact, then $\exists f\in C_c(G)\cap P$ (function of positive types)

s.t. $\hat f\ge 0$ on $\hat G$, and $\hat f>0$ on G

proof: We only need to consider open neighborhood (then by finite covering (K compact))

Pick $h\in C_c(G)$, with $\hat h(1) = \int h = 1$, and set $g = h^* * h$ (will found transform, also g is a function of positive type.)

then g is also in $C_c(G)$, $\hat g(\xi) = |\hat h(\xi)|^2\ge 0$

and $\hat g|_V>0$, for some open neighborhood V of e (identity)

then $\exists\xi_1,\dots,\xi_n$ s.t. $K\subset\bigcup_1^n\xi_i^{-1}V$ (Finite covering), and take $f = (\sum_{j=1}^n\xi_j)g$

$\hat f$ 依赖 g $\Rightarrow$ $\hat g(\cdot\,\xi_j)$，对于 $\xi_j^{-1}V$ ∎

Lemma 4.21: If $f,g\in B^1$, then $\hat f\,d\mu_g = \hat g\,d\mu_f$

proof: Recall that $\phi(x) = \int\langle x,\xi\rangle d\mu_\phi(\xi)$

$\forall h\in L^1(G)$, then $\int\hat h\,d\mu_f \overset{\text{by def}}{=} \iint\langle x^{-1},\xi\rangle h(x)dx\,d\mu_f(\xi)$

$= \int f(x^{-1})h(x)dx$

$= f*h(1)$, now we use properties of convolution.

therefore $\int\hat g\,\hat h\,d\mu_f$

$= \int\widehat{g*h}\,d\mu_f = g*h*f(1)$

$= f*h*g(1) = \int\hat f\,\hat h\,d\mu_g$

As $\{\hat h: h\in L^1(G)\}$ is dense in $C_0(\hat G)$, we have $\hat g\,d\mu_f = \hat f\,d\mu_g$ ∎

With Lemma 4.20 and Lemma 4.21, we can now prove thm 4.22

<u>proof</u> of thm 4.22: We need to use the uniqueness of Haar measure $d\xi$.

$\forall \psi \in C_c(\hat{G})$, by lemma 4.20, $\exists\, g \in U$, s.t. $\hat{g} > 0$ on $\operatorname{supp}\psi$, then define

functional

$$I(\psi) = \int \frac{\psi}{\underbrace{\hat{g}}_{>0}}\, d\mu_g$$

, now we want to show I is irrelevant of g using lemma 4.21

Now we show $I(\psi)$ is independent of g (only depends on ψ)

$\forall f$, $\hat{f} > 0$ on $\operatorname{supp}\psi$

$$I(\psi) = \int \frac{\psi}{\hat{g}}\, d\mu_g = \int \frac{\psi}{\hat{f}\hat{g}} \boxed{\hat{f}\, d\mu_g} \overset{\text{lemma 4.21}}{=} \int \frac{\psi}{\hat{f}\hat{g}} \boxed{\hat{g}\, d\mu_f} = \int \frac{\psi}{\hat{f}}\, d\mu_f$$

$\downarrow$

bounded linear functional on $C_c(\hat{G})$

Next, we show left invariance of I, i.e. $\boxed{I(L_\eta \psi) = I(\psi)}$, $\forall \eta \in \hat{G}$

By def, now $\int \langle x, \xi\rangle \boxed{(L_\eta)_* d\mu_g} \overset{\text{def}}{=} \int \langle x, \eta^{-1}\xi\rangle\, d\mu_g(\xi)$

可以拿出

$$= \langle x, \eta^{-1}\rangle\, g(x) \overset{\text{def}}{=} \int \langle x, \xi\rangle \boxed{d\mu_{\eta^{-1}g}(\xi)}$$

now by uniqueness (4.18), we have that $(L_\eta)_* d\mu_g = d\mu_{\eta^{-1}g}$

Therefore $I(L_\eta \psi) = \int \frac{\psi(\eta^{-1}\xi)}{\hat{g}(\xi)}\, d\mu_g(\xi)$

$$= \int \frac{\psi(\eta^{-1}\xi)}{\widehat{\eta^{-1}g}(\eta^{-1}\xi)}$$

, As $\widehat{\eta^{-1}g}(\eta^{-1}\xi) \overset{\text{def}}{=} \int \overline{\langle x, \eta^{-1}\xi\rangle}\, \eta^{-1}(x)\, g(x)\, dx$

$= \hat{g}(\xi)$

$$\text{then} = \int \frac{\psi(\xi)}{\widehat{\eta^{-1}g}(\xi)} (L_\eta)_* d\mu_g(\xi) = \int \frac{\psi(\xi)}{\widehat{\eta^{-1}g}(\xi)}\, d\mu_{\eta^{-1}g}(\xi)$$

$$= I(\psi).$$

Therefore $I(\psi)$ induces a left-invariant measure on $\hat{G}$, thus must be a Haar measure, by uniqueness

$$I(\psi) = \int \psi(\xi)\, d\xi \Rightarrow I(\psi \hat{f}) = \int \psi \hat{f} \quad , \forall \psi \in C_c(\hat{G})$$

$$\Vert \int \frac{\psi \hat{f}}{\hat{f}}\, d\mu_f = \int \psi\, d\mu_f$$

$$\Rightarrow d\mu_f = \hat{f}(\xi)\, d\xi$$

$$\Rightarrow f(x) = \int \langle x, \xi\rangle\, d\mu_f(\xi) = \int \langle x, \xi\rangle \hat{f}(\xi)\, d\xi \qquad \blacksquare$$

Lecture 22: Pontrjagin Duality

Lecture 22-2023 年 5 月 23 日下雨 ☂ 24°C-29°C

主要内容：

其他信息：

还有三节课. Final 方床习回 → take home 2~3h ; midterm 10 · Final 10 → 还没给 ; 80分+ ; min 86门 ↑

Remark: Last lecture we've shown that $\mu \in M(G)$ is invertible iff $\hat{\mu}$ has a zero

and if μ is not invertible $\Rightarrow \mu \in J$, maximal ideal (closed)

then $M(G)/J$ is a Banach Algebra

every element in $M(G)/J$ is invertible, due to maximality of J

$\Rightarrow M(G)/J \cong \mathbb{C}$ (if $\exists\, x \neq \lambda e$, then $(\lambda e - x)^{-1}$ is entire).

prop 4.18: If $\mu \in M(\hat{G})$, $\phi_\mu(x) = \int \langle x, \xi\rangle \, d\mu(\xi)$, then $\phi_\mu = 0 \Rightarrow \mu = 0$.

Recall def: $B(G) = \{\phi_\mu : \mu \in M(\hat{G})\}$ = linear span of $P(G)$

and $B'(G) = B(G) \cap L'(G)$, in particular, if $f \in L'$, then $f^* * f \in B'(G)$

很多定理都只在 $B'(G)$ 上证

and the first version of Fourier Inversion.

Thm 4.22: (Fourier Inversion thm I)

If $f \in B'$, then $\hat{f} \in L'(\hat{G})$, and $f(x) = \int \langle x, \xi\rangle \hat{f}(\xi)\, d\xi$ ← suitably normalized (某个常数), called dual measure.

$(\Leftrightarrow d\mu_f(\xi) = \hat{f}(\xi)\, d\xi)$

Example of dual measure.

① probability Haar measure on compact G ($\hat{G}$) $\leftrightarrow$ Counting measure on discrete $\hat{G}$ (G)

② $\mathbb{R}$ is self-dual by the pairing $\langle x, \xi\rangle = e^{2\pi i x \cdot \xi}$

③ $\mathbb{Q}_p$ is self-dual. moreover $\widehat{\chi_{B(1,0)}} = \chi_{B(1,0)}$ ← not possible in $\mathbb{R}$ case. the reason why we prefer $\mathbb{Q}_p$ in number theory (one of)

$B(1,0) = \mathbb{Z}_p$, integer ring, subgroup, compact

$\parallel$

$\sum_{j=0}^{\infty} c_j p^j$, with $\{1, \cdots, p-1\}^{\mathbb{N}}$ topology (product)

Every character $\xi_y \in \hat{\mathbb{Q}_p} \cong \mathbb{Q}_p$, can be restricted onto $\mathbb{Z}_p$, that gives $\xi_y{}' \in \hat{\mathbb{Z}_p}$

$$\widehat{\chi_{B(1,0)}}(\xi_y) = \int 1 \cdot \bar{\xi_y}\, |\chi_{B(1,0)} = \langle 1, \xi_y{}'\rangle_{\mathbb{Z}_p} = \begin{cases} 1, & \xi_y|_{\mathbb{Z}_p} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$\xi_y(x) = e^{2\pi i y \cdot x} \Leftrightarrow y \in \mathbb{Z}_p$

即 $\chi_{B(1,0)}$.

(see Rudin, 1.5.2)

Now we consider an important Corollary of Thm 4.22 "characters separate points" ← used later in the proof of pontrjagin duality

$\Downarrow$

$\forall x, y \in G$, $\exists\, \xi \in \hat{G}$, $\xi(x) \neq \xi(y)$.

It suffices to show: $\forall x_0 \in G \backslash \{e\}$, $\exists\, \xi \in \hat{G}$ s.t. $\xi(x_0) \neq 1$

$\exists$ symmetric neighborhood V of e s.t. $x_0 \in V \cdot V$, then take $g(x) = \chi_V * \chi_V^* \in B'$

By taking 4.22. $g(x_0) = \int \langle x_0, \xi\rangle |\widehat{\chi_V}(\xi)|^2 d\xi$

$\parallel$ 0

拨有 >0 if $\langle x_0,\xi\rangle = 1$, $\forall \xi \in \widehat{G}$

contradiction. done.

Also a corollary of thm 4.22

Thm 4.26 (plancherel)

The Fourier transform on $L^1 \cap L^2$ extends uniquely to an isometry between $L^2(G)$ and $L^2(\widehat{G})$

proof: $\forall f \in L^1 \cap L^2$, $f^* * f \subset \mathcal{B}^1$ (involution. Banach $*$-algebra), then $|\hat{f}|^2 \in L^1$ (by thm 4.22).

and $\int |f|^2 = f^* * f(1) \underset{\text{thm } 4.22}{=} \int |\hat{f}(\xi)|^2 d\xi$

Note that $L^1 \cap L^2$ is dense in L^2. Now we see $f \mapsto \hat{f}$ extends to $L^2(G) \to \mathcal{F}(L^2(G)) \subset L^2(\widehat{G})$

It remains to show it is onto

If $\exists \psi \in L^2(\widehat{G})$, s.t. $\int \psi \cdot \hat{f} = 0$, $\forall f \in L^1 \cap L^2$, then $\forall x$. $\int \psi \, \widehat{L_x f} = 0$

$$\parallel \quad \int \langle x^{-1}, \xi\rangle \underbrace{\psi(\xi) \hat{f}(\xi)}_{\in L^1(\widehat{G})} d\xi$$

$\int \langle y,\xi\rangle f(x^{-1}y)dy = \langle x,\xi\rangle \hat{f}(\xi)$?

$\Rightarrow \psi(\xi)\hat{f}(\xi) = 0$. a.e. by prev result

可换成 $\forall$ C_0 中的

$\Rightarrow \psi(\xi) = 0$, a.e. as $\mathcal{F}(L^1)$ is dense in C_0.

$\Downarrow$

Surjective. ▯

Corollary: If G is compact, then $\widehat{G}$ is an orthonormal basis for $L^2(G)$

4.3: The Pontrjagin Duality Thm

$\Phi: G \hookrightarrow \widehat{\widehat{G}}$, $\langle \Phi(x), \xi\rangle = \langle x, \xi\rangle$

Thm 4.3.2: Φ is an isomorphism between topological groups. {群同构, 拓扑同构}

结构定理

Now assuming thm 4.32

Thm 4.33 (Fourier Inversion Theorem II)

If $f \in L^1$, $\hat{f} \in L^1 \Rightarrow f(x) = \hat{\hat{f}}(x^{-1})$ a.e. (or equivalently $f(x) = \int \langle x, \xi\rangle \hat{f}(\xi) d\xi$ a.e.)

In particular. "=" hold every where if f is continuous. measure theory: $f=0$ a.e., f continuous $\Rightarrow f=0$ everywhere.

proof: by def $\hat{f}(\xi) = \int \overline{\langle x,\xi\rangle} f(x)dx = \int_{G = (\widehat{G})^\wedge} \langle x,\xi\rangle \boxed{f(x^{-1})dx}$ see as $f(x^{-1})dx \in M(\widehat{\widehat{G}})$

by Pontrjagin duality, we could use thm 4.22

Recall thm 4.22, $f \in \mathcal{B}^1(G)$, then $f(x) = \int_{\widehat{G}} \langle x,\xi\rangle \hat{f}(\xi) d\xi$

$\parallel$

$L^1 \cap \{\int_{\widehat{G}} \langle x,\xi\rangle d\mu(\xi),\ \mu \in M(\widehat{G})\}$

Thus theorem 4.22 applies to $\hat{f} \in \mathcal{B}(\widehat{G})$, and $\hat{f}(\xi) = \int \langle x,\xi\rangle \hat{\hat{f}}(x)dx$

$\Rightarrow \int_{(\widehat{G})^\wedge} \langle x,\xi\rangle \left(f(x^{-1}) - \hat{\hat{f}}(x)\right) dx = 0$, $\forall \xi \in \widehat{G}$

Recall prop 4.18. $\mu \in M(\widehat{G})$. $\phi_\mu(x) = \int_{\widehat{G}} \langle x,\xi\rangle d\mu(\xi) = 0 \Rightarrow \mu = 0$

Now by prop 4.18, $f(x^{-1}) = \hat{\hat{f}}(x)$ a.e. $\square$

uniqueness of Fourier transform

Corollary 4.34: $\hat{\mu} = \hat{\nu}$, $\mu, \nu \in M(G) \Rightarrow \mu = \nu$

proof: By Pontryagin duality. $\hat{\mu} = \hat{\nu} \Leftrightarrow \phi_\mu = \phi_\nu$ (take $\alpha \in \hat{\hat{G}}$

$\xRightarrow{\text{Prop 4.18}} \mu = \nu$

Cor 4.36: G is compact/discrete $\Leftrightarrow$ $\hat{G}$ is discrete / compact

现在是双向的
之前是单向

Cor: $L^1(G)$ has multiplicative identity $\Leftrightarrow$ G is discrete.

Now we go back to the proof of thm 4.32 (Pontryagin), we need 2 lemmas

h is the "Fourier inverse" of $\phi * \psi$

Lemma 4.30: If $\phi, \psi \in C_c(\hat{G})$, then $\phi * \psi = \hat{h}$, for some $h \in B^1(G)$.

In particular $\mathcal{F}(B^1)$ is dense in $L^p(\hat{G})$, for $p < \infty$ — we only use the case $p=2$, later.

proof: let $f(x) = \int \langle x, \xi \rangle \phi(\xi)\, d\xi$, $g(x) = \int \langle x, \xi \rangle \psi(\xi)\, d\xi$.

later we will show that $h \in L^1(G)$

$$h(x) \stackrel{\text{def}}{=} \underbrace{\int \langle x, \xi \rangle \phi * \psi(\xi)\, d\xi}_{\text{of the form } \in B(G)} \boxed{\in B^1(G)}$$

$$= \iint \langle x, \xi \rangle \phi(\xi \eta^{-1}) \psi(\eta)\, d\eta\, d\xi$$

$$= \iint \langle x, \xi\eta \rangle \phi(\xi) \psi(\eta)\, d\eta\, d\xi$$

$$= f(x) \cdot g(x)$$

then $f, g, h \in B$.

利用延拓 由$L^2 \to L^1$的结果, we shall show $f, g \in L^2$, so $h \in L^1$

$\forall k \in L^1 \cap L^2(G)$ (test function, dense in L^2 $\Rightarrow$ extend到 L^2)

$$\int f \bar{k} = \iint \langle x, \xi \rangle \phi(\xi)\, d\xi\, \bar{k}(x)\, dx$$

$$\stackrel{\text{Fubini}}{=} \int \phi \cdot \overline{\hat{k}} \stackrel{\text{Cauchy-Schwarz}}{\leq} \|\phi\|_{L^2} \cdot \|\hat{k}\|_{L^2} \stackrel{\text{Plancherel}}{=} \|\phi\|_{L^2} \cdot \|k\|_{L^2}$$

$\Rightarrow f \in L^2$, similarly for g. so $h \in L^1(G)$. Now $\boxed{h \in B^1(G)}$

So by thm 4.22

$$h(x) = \int_{\hat{G}} \langle x, \xi \rangle \hat{h}(\xi)\, d\xi \stackrel{\text{2个表达式}}{=} \int_{\hat{G}} \langle x, \xi \rangle \phi * \psi(\xi)\, d\xi$$

by prop 4.18 $\hat{h}(\xi) = \phi * \psi(\xi)$.

Finally $\mathcal{F}(B^1)$ is dense in L^p, as $\{\phi * \psi,\ \phi, \psi \in C_c(\hat{G})\}$ is dense in L^p. $\square$

pure pt topology

no need to be abelian

Lemma 4.31 : Suppose G is locally cpt, and $H \leq G$, a subgp. If H is locally cpt in the relative topology then H is closed.

used later

in embedding in proof of Pontrjagin

surjective to Im is closed.

Next time Lemma 4.31. pontrjagin duality. Corollary. Bohr compact

June 10^{th} ~ take home exam. (after the final week).

Always searchable in stack exchange

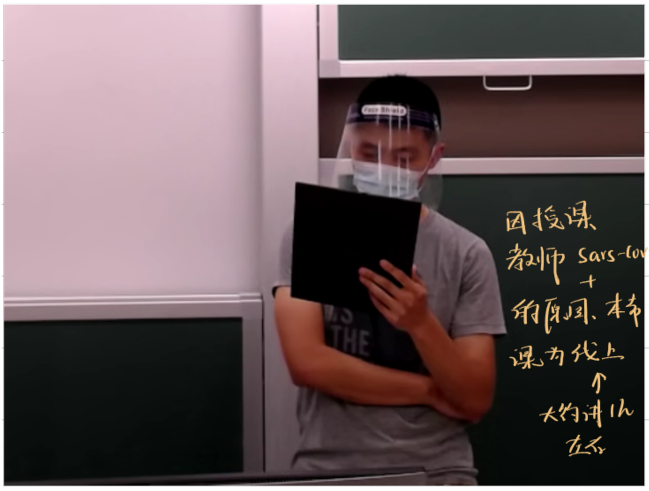
Lecture 23: Proof of Pontrjagin Duality, Poisson Summation Formula

Lecture 23-2023 年 5 月 25 日下雨 ☂ 24°C-29°C

主要内容：利用两个 Lemma 来证明 Pontrjagin duality 定理

下节课证明 Theorem 4.40，注意理解第二个 isomorphism Ψ，这个定理后面会被用来证明 Poisson Summation Formula

其他信息：老师阳了，转线上



cont. Pontryagin Duality

$\Phi : G \to \widehat{G}$ ← isomorphism

$x \quad \langle \Phi(x), \xi \rangle = \langle x, \xi \rangle$

and last time, we have shown

Lemma 4.30: If $\phi, \psi \in C_c(\widehat{G})$, then $\phi * \psi = \widehat{h}$ for some $h \in B'(G)$.

In particular $\mathcal{F} \subset B'(G)$, is dense in $L^P(\widehat{G})$

Lemma 4.31 (点集拓扑) Suppose G is a locally cpt group, and H is a subgroup.

If H is locally compact in the relative topology, then H is closed.

Proof: Notice that every relative cpt subset of H must be compact in G. So "H is locally compact in the relative topology" means that $\forall$ relative neighborhood of 1, $\exists\ U \cap H$ (open in G), where relative closure $K \subset H$ is a cpt subset of G

$\Downarrow$

(that implies that the closure of $U \cap H$ in G is a subset of H.)

$\exists$ symmetric neighborhood V of 1, s.t. $V \cdot V \subset U$

Say we have a net (sequence) $\underset{\in H}{x_\alpha} \to x$ in G, we need to show that $x \in H$.

since $x \in \overline{H}$ (group), so is x^{-1}, pick $y \in Vx^{-1} \cap H$. 用y，因为x不知道x∈H？所以用y把x拖到H旁边.

Eventually x_α lies in $x \cdot V$, so $\underset{\in H}{y \cdot x_\alpha}$ lies in $Vx^{-1}xV = V \cdot V \subset U$

As $\overline{H \cap U} \subset H$, and $y x_\alpha \to y x \Rightarrow yx \in H \Rightarrow x \in H$. ▥

With the above 2 lemmas, we could prove the Pontryagin Duality

Proof: As characters separate points, i.e. $\forall x \neq e$, $\exists \xi \in \widehat{G}$, s.t. $\langle x, \xi \rangle \neq 1$

$\iff \forall x \neq y$, $\exists \xi \in \widehat{G}$, s.t. $\langle x, \xi \rangle \neq \langle y, \xi \rangle$)

So Φ is 1-1 (onto)

Next, we show that $\Phi : G \to \Phi(G) \subset \widehat{\widehat{G}}$ is a homeomorphism, by showing

suppose $x \in G$, and $\{x_\alpha\}_{\alpha \in A}$ is a net in G, then (i) → (iv) are equivalent

① $x_\alpha \to x$ in G

② $f(x_\alpha) \to f(x)$, $\forall f \in B'(G)$

③ $\int \langle x_\alpha, \xi \rangle \check{f}(\xi)\, d\xi \to \int \langle x, \xi \rangle \check{f}(\xi)\, d\xi$, $\forall f \in B'(G)$

④ $\Phi(x_\alpha) \to \Phi(x)$ in $\widehat{\widehat{G}}$

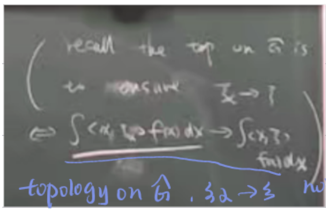
trivial.

① $\Rightarrow$ ② $\Longleftrightarrow$ ③ $\Longleftrightarrow$ ④ → $\mathcal{F}(B')$ is dense in $L^1(\widehat{G})$, lemma 4.30

(4)→(3) by DCT (dominated convergence thm)

by $\boxed{\text{thm } 4.22}$ $f \in B'$, then $f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$. $\hat{f} \in L^1$

(3)→(4) needs more explanation.



Since $\mathcal{F}(B')$ is dense in $L^1(G)$

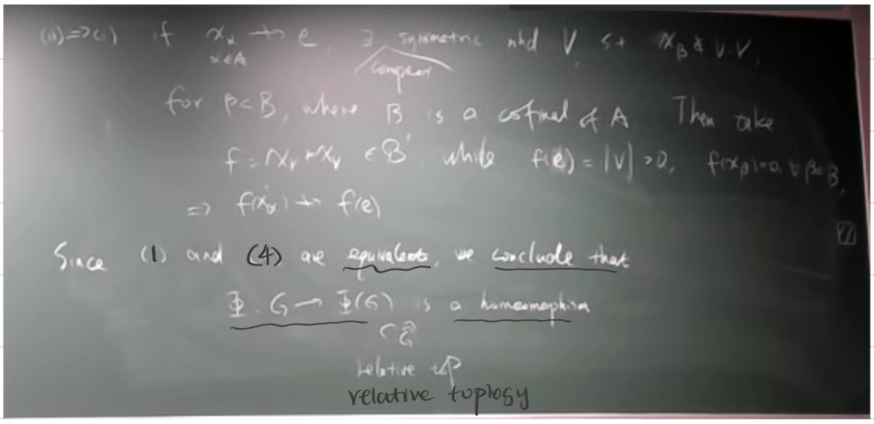
(iii) $\Rightarrow \int \langle x_\alpha, \xi \rangle g(\xi) d\xi \to \int \langle x, \xi \rangle g(\xi) d\xi$, $\forall g \in L^1(\widehat{G})$

$\Rightarrow x_\alpha \to x$ on $\widehat{\widehat{G}}$

Now ② $\Rightarrow$ ①: if $x_\alpha \not\to e$ ($\alpha \in A$), $\exists$ compact symmetric neighborhood $x_\beta \notin V \cdot V$ for $\beta \in B$, where β is a cofinal of A, then take

$f = \chi_V \cdot \chi_V \in B'$, while $f(e) = |V| > 0$, $f(x_\beta) = 0$, $\forall \beta \in B$.

$\Rightarrow f(x_\alpha) \not\to f(e)$. ▨



As G is locally cpt, so is $\Phi(G)$, thus closed in $\widehat{\widehat{G}}$ by lemma 4.31

We shall show $\Phi(G) = \widehat{\widehat{G}}$, If otherwise, $\exists x \in \widehat{\widehat{G}} \setminus \underbrace{\Phi(G)}_{\text{closed}}$

$\exists$ symmetric compact neighborhood V of e s.t. $xV \cdot V \cap \Phi(G) = \emptyset$

then if we take $\phi \in C_c(x \cdot V)$, $\psi \in C_c(V)$, positive, then

$\phi * \psi |_{\Phi(G)} = 0$

$\|$ Lemma 4.30

$\hat{h}$ for some $h \in B'(\widehat{G})$

Therefore $0 = \phi * \psi(\Phi(x^{-1})) = \hat{h}(\Phi(x^{-1}))$

$= \int \langle \Phi(x), \xi \rangle h(\xi) d\xi$

$= \int_{\widehat{G}} \langle x, \xi \rangle h(\xi) d\xi$, $\forall x \in G$

Prop 4.18 $\Rightarrow h = 0$ a.e.

$\Rightarrow \hat{h} = 0 \Rightarrow \phi * \psi = 0$, contradiction. $\Rightarrow \Phi(G) = \widehat{\widehat{G}}$ ▨ Pontrjagin duality

Now Prop 4.37: If $f, g \in L^2(G)$, then $(fg)^\wedge = \hat{f} * \hat{g}$

we use schwarz functions in $\mathbb{R}$ case, here we use B' functions (dense)

Proof: It suffices to assume $f, g \in L^2(G) \cap \underset{\text{dense in } L^2(G)}{\mathcal{F}(B'(G))}$

then $\exists\ \phi, \psi \in B'(G) \cap L^2$, s.t. $f(x) = \hat{\phi}(x^{-1})$, $g(x) = \hat{\psi}(x^{-1})$

By Fourier Inversion thm 1 (thm 4.22)

$$\hat{f}(\xi) = \int \overline{\langle x, \xi \rangle}\, \hat{\phi}(x^{-1})\, dx = \underbrace{\int \langle x, \xi \rangle \hat{\phi}(x)\, dx}_{\overset{\text{thm } 4.22}{\|}\ \phi(\xi)}$$

Similarly $\hat{g} = \psi$

$\Rightarrow \hat{f} * \hat{g} = \phi * \psi$ = RHS

On the LHS: $fg = \hat{\phi}(x^{-1}) \cdot \hat{\psi}(x^{-1})$, by def

$= \widehat{\phi * \psi}(x^{-1}) \in L^1$

Now, by Fourier Inversion $(fg)^\wedge = \phi * \psi(x)$.

Finally, apply the density argument. $\square$

Now, at the end, we need to explain Poisson Summation Formula

$f \in L^1(\mathbb{R})$, $\hat{f} \in L^1(\mathbb{R})$, then $\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i x \cdot n}$ In Euclidean case

in particular $\sum_{n \in \mathbb{Z}} \underset{\substack{\text{dual lattice} \\ \text{physical}}}{f(n)} = \sum_{n \in \mathbb{Z}} \underset{\text{freq space}}{\hat{f}(n)}$, 整给 x

to General group

take $\mathbb{Z} \overset{\text{subgp}}{\leq} \mathbb{R}$ $\Rightarrow$ Integral on subgp

Def: If H is a closed subgroup of G, define

$H^\perp = \{\xi \in \hat{G} : \langle x, \xi \rangle = 1,\ \forall x \in H\}$, a closed subgp of $\hat{G}$

Denote $q: G \to \underset{\substack{\text{locally cpt} \\ \text{abelian}}}{G/H}$, natural projection.

abelian, 그래서 closed 가 Hausdorff (why we need closed)

Prop 4.39: $(H^\perp)^\perp = H$

Proof: $H \subset \underset{\leq G}{(H^\perp)^\perp}$ by definition. Conversely $\forall x_0 \notin H$, $q(x_0) \neq e$ in G/H

$\Downarrow$

$\exists\ \eta \in (G/H)^\wedge$ s.t. $\eta(q(x_0)) \neq 1$

$\Downarrow$

It can be lifted to a character of G

$\xi(x) = \eta \circ q(x)$

$\Downarrow$

$\xi = 1$ on H, $\xi(x_0) = \eta \circ q(x_0) \neq 1$

$\Rightarrow \exists\ \xi \in H^\perp$ s.t. $\xi(x_0) \neq 1 \Rightarrow x_0 \notin (H^\perp)^\perp \Rightarrow (H^\perp)^\perp \subset H$

$\square$

Prove next time

Thm 4.40: Suppose H is a closed subgp of G, then

With this

$\Phi: (G/H)^\wedge \to H^\perp$, $\psi: \hat{G}/H^\perp \to \hat{H}$

We could

$\phi(\eta) = \eta \circ q$ $\qquad$ $\psi(\xi H^\perp) \mapsto \xi|_H$

prove the

Φ, Ψ are isomorphism of topological group.

Poisson summation formula.

代数&拓扑 continuity

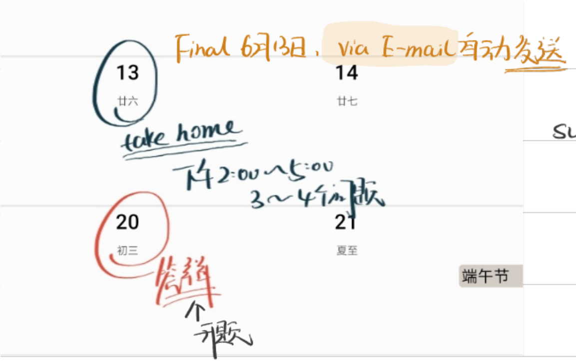
Lecture 24: cont. Poisson Summation Formula

Lecture 24-2023 年 6 月 1 日昨晚下雨，今日没那么热 27°C-34°C

主要内容：首先证明了 Theorem 4.40，随后用其中的 isomorphism 关系证明了 Poisson summation formula（其实是 Fourier inversion 的一种推广）；最后是 Bohr 紧化，与 almost periodic function 有很大的关系（Tao 的工作）

本课程的最后一个 Lecture

其他信息：6 月 13 日下午 2 点到 5 点，Take-home exam



summer school ~16篇左右

Recall poisson summation formula in $\mathbb{R}$: $\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i x n}$

Def: H closed subgroup of G, define $H^\perp \overset{\text{def}}{=} \{\xi \in \hat{G},\ \xi|_H = 1\}$

Prop 4.39: $(H^\perp)^\perp = H$

Natural projection: $q: G \to G/H$,

$$\begin{tikzcd} G \arrow[d, "q"'] \arrow[dr, dashed, "\eta\circ q"] & \\ G/H \arrow[r, "\eta"'] & S^1 \end{tikzcd}$$

提升, 投影, 限制

only this one is used later

Thm 4.40: $\Phi: (G/H)^\wedge \to H^\perp$: $\Phi(\eta) = \eta\circ q$

$\psi: \hat{G}/H^\perp \to \hat{H}$: $\psi(\xi\cdot H^\perp) = \xi|_H$

are isomorphisms between topological group.

Proof: Group isomorphism ✓

We first show that Φ is continuous, say $\eta_\alpha \to \eta$ in $(G/H)^\wedge$

当时定义 characters 收敛的一个等价形式

$\Leftrightarrow \eta_\alpha \to \eta$ in every compact subset K of G/H

$\Rightarrow \eta_\alpha\circ q \to \eta\circ q$ uniformly in every compact subset of G

$\Rightarrow \eta_\alpha\circ q \to \eta\circ q$

Conversely, if $\eta_\alpha\circ q \to \eta\circ q$, By Lemma 2.48, $\forall$ compact $F\subset G/H$, $\exists$ a compact $K\subset G$, s.t $q(K)=F$

So $\eta_\alpha\circ q \to \eta\circ q$ uniformly on $K \Rightarrow \eta_\alpha \to \eta$ uniformly on F.

Hence Φ is an isomorphism.

By Φ is an isomorphism.

Now, we turn to ψ. By Φ $(\hat{G}/H^\perp)^\wedge \cong (H^\perp)^\perp \cong H$

$\Leftrightarrow (\hat{G}/H^\perp)^{\wedge\wedge} \cong H^\wedge$

$\|$ duality

$\hat{G}/H^\perp \cong \hat{H}$, $\Rightarrow$ Done.

More precisely, $\forall x\in H$, its corresponding element $y \in (\hat{G}/H^\perp)^\wedge$ is given by

$\langle y, \xi H^\perp\rangle = \langle x, \xi\rangle$, $\forall \xi\in\hat{G}$

With these isomorphisms, we could calculate some dual group (used to be hard to calculate)

Example: $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ (subgp) , $\xi_y(x) = e^{2\pi i x \cdot y}$, $\xi_1(x) = e^{2\pi i \alpha}$

observe that $\mathbb{Z}_p^\perp = \mathbb{Z}_p = \{\sum_{j=0}^{\infty} c_j p^j, c_j = 0,1,2,\dots\}$

So by thm 4.40

$$\widehat{\mathbb{Z}_p} \cong \mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Q}_p/\ker \xi_1 \cong \text{Range}(\xi_1) = \{p^k\text{-th root of the unity}, k=1,2,\dots\}$$

$$\mathbb{Z}_p = \sum_{j=-N}^{-1} c_j p^j$$

As $\mathbb{Z}_p$ is compact, $\widehat{\mathbb{Z}_p}$ is discrete

overall, $\widehat{\mathbb{Z}_p} = \{p^k\text{-th root of unity}\}_{k=1,2,\dots}$ with discrete topology.

Now we could prove the Poisson summation Formula

A generalization of Fourier Inversion. see remark below

Thm 4.43: $f \in C_c(G)$, define $F \in C_c(G/H)$ by

$$F(xH) = \int_H f(xy)\,dy, \text{ then}$$

$$\hat{F} = \hat{f}|_{H^\perp}, \text{ where we identify } H^\perp \cong (G/H)^\wedge_{4.40}$$

If also, $\hat{f}|_{H^\perp} \in L^1(H^\perp)$, then

$$\int_H f(xy)\,dy = \int_{H^\perp} \hat{f}(\xi)\langle x,\xi\rangle\,d\xi.$$

中间的步骤, non-trivial.

Remark: ① If $H = \{e\}$, just Fourier Inversion

② If $G = \mathbb{R}$, $H = \mathbb{Z}$, $H^\perp = \mathbb{Z}$ $\Rightarrow$ classical poisson summation formula on $\mathbb{R}$

use Fourier Inversion

Proof: $\forall \xi \in H^\perp = (G/H)^\wedge$, $\langle xy,\xi\rangle = \langle x,\xi\rangle$, $\forall y \in H$

$$\hat{F}(\xi) = \int_{G/H} \left(\int_H f(xy)\,dy\right) \overline{\langle xH,\xi\rangle}\,dxH$$

$\langle xH,\xi\rangle = \langle xy,\xi\rangle$, $\forall y \in H$ — Here we implicitly use $H^\perp \cong (G/H)^\wedge$

$$= \int_{G/H}\int_H f(xy)\,\overline{\langle xy,\xi\rangle}\,dy\,dxH$$

对 coset 积分 + 公式, thm 2.51

$$= \int_G f(x)\,\overline{\langle x,\xi\rangle}\,dx = \hat{f}(\xi)$$

Finally, if $\hat{f}|_{H^\perp} \in L^1$ ($\Leftrightarrow \hat{F} \in L^1((G/H)^\wedge)$), then apply Fourier Inversion to $\hat{F}$ on $(G/H)^\wedge \cong H^\perp$

$$F(x) = \int_{(G/H)^\wedge} \hat{F}(\xi)\langle x,\xi\rangle\,d\xi = \int_{H^\perp} \hat{f}(\xi)\langle x,\xi\rangle\,d\xi$$

$$F(x) = \int_H f(xy)\,dy.$$

▥

Now, the final section of this part

4.7: Bohr Compactification (non-compact G)

$G \to \widehat{G} \to \widehat{G}_d \to (\widehat{G}_d)^\wedge \overset{def}{=} bG$

character; $\widehat{G}_d$: discrete topology (cpt群); $(\widehat{G}_d)^\wedge$: all gp homomorphisms $\widehat{G} \to S^1$ (no assumption on continuity) 由discrete topology

$\widehat{\widehat{G}} = \{$all continuous group homomorphism $\widehat{G} \to S^1\}$

$\cong$ $G \hookrightarrow bG$ as a subgroup.

Now we show that G is dense in bG. Consider $\overline{G} \subset bG$ (closed subgp)

$$(\overline{G})^\perp = \{\text{characters on } G \text{ that is trivial on } G\} = \{1\}$$

$\cap$

$\widehat{G}_d = \{$charcters on G with discrete topology$\}$

$\Longrightarrow$ 由 charcters separate pts. $\overline{G} = [(\overline{G})^\perp]^\perp =$ the whole group bG.

$\Downarrow$ dense

Also, the embedding $G \hookrightarrow bG$ is continuous

$x_\alpha \to x$ in $G \Leftrightarrow x_\alpha(\xi) \to x(\xi)$, on every cpt subset of $\widehat{G}$ — compact convergence topology

stronger $\Rightarrow$ embedding is continuous

and $x_\alpha \to x$ in $bG \Leftrightarrow x_\alpha(\xi) \to x(\xi)$ pointwise on $\widehat{G}_d$

discrete topology; cpt ⇒有限点 ⇒ 单点

However the embedding is not a homeomorphism.

If so, by lemma 4.31, G must be closed, thus cpt $\Rightarrow$ contradiction!

closed subgp of cpt

Hard to give Example of Bohr compactification

more complicated than "single pt compactification"

Prop 4.80: If K is a compact group, and $\rho: G \to K$, a continuous homomorphism then ρ extends to a continuous homomorphism from bG to K

$$\begin{tikzcd} bG \arrow[dr, dashed] & \\ G \arrow[u] \arrow[r, "\rho"] & K \end{tikzcd}$$

e.g. $K = S^1$

proof: We may assume $K = \overline{\rho(G)}$, Abelian group, then ρ^* (adjoint) $: \widehat{K} \to \widehat{G}$, $\rho^*(\eta) = \eta \circ \rho$

由 K cpt $\Rightarrow$ discrete topology $\Rightarrow \rho^*$ is continuous from $\widehat{K}$ to $\widehat{G}_d$

take adjoint again to get a continuous group homomorphism from bG to K. $\boxed{///}$

Bohr compactification $\Longrightarrow$ used to study almost periodic function.

Def. A bounded continuous function f on G is called uniformly almost periodic, if the set of translates of f, $\{R_x f, x \in G\}$, is totally bounded in the uniform metric

$\Updownarrow$ Recall

($\forall \varepsilon > 0$ $\exists x_1, \dots, x_n \in G$, s.t. $\forall x \in G$, $\exists x_j$ s.t.

$\|R_x f - R_{x_j} f\|_\infty < \varepsilon$)

$\| \|R_{x_j^{-1}x} f - f\|_\infty$

almost periodic.

Remark: $x_j^{-1} x$ may be large, e.g. $\mathbb{R}$.

Property: Such an f must be uniformly continuous: consider $K \overset{\text{def}}{=} \overline{\{R_x f, x \in G\}}$, compact

If f is not uniformly continuous,

$\exists x_\alpha \to e \in G$, s.t. no subsequence of $R_x f$ is uniformly convergent to f

Since K is compact, $\exists$ a subsequence but $R_{x_\alpha} f \to f$ pointwise (由于 $\|\cdot\|_\infty$)

$R_{x_\alpha} f \to g$ uniformly, contradiction, as it applies to $g = f$.

Thm 4.81: If f is bounded continuous function on G, TFAE

more intuitive

① f is the restriction to G of a continuous function on bG

② f is the uniform limit of linear combinations of characters on G

相当于 exponentials sum 的极限

③ f is uniformly almost periodic.

proof: ①⇒② By Stone-Weierstrass, linear combinations of characters of bG (compact group) are uniformly dense in $C(bG)$

So the extension of f can be approximated by characters of bG, then restrict everything to G.

characters 也是 continuous function on bG

②⇒① : By prop 4.80, the sequence on G can be extended to a sequence on bG

Also uniform convergence on a dense subset ⇒ uniform convergence

①⇔③ is a little tricky (due to the definition of almost periodic)

①⇒③ Say $f = \phi|_G$, $\phi \in C(\underbrace{bG}_{\text{cpt}})$

Since $x \mapsto R_x f$ is continuous.

$\cap$ bG cpt, $\cap$ $C(bG)$

the set $\{R_x f, x \in bG\}$ is cpt in $C(bG)$, that has dense subset $\{R_x f, x \in G\}$, thus totally bounded

③ ⇒ ①: Take $K = \overline{\{R_x f,\ x \in G\}}$, compact (by almost periodic) ↗ totally bdd.

↗ isometric bijection

By Arzela-Ascoli, $Iso(K)$ is a compact group

Notice $R: \underset{G}{\overset{\cap}{x}} \longmapsto \underset{Iso(K)}{\overset{\cap}{R_x}}$ continuous gp homomorphism

So it can be extended to $\tilde{R}: bG \to Iso(K)$

then $f = \tilde{R}_x f(1)|_G$, where $x \in bG$. $\boxed{///}$

本笔记条目索引

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[Folland-Real] Folland G.B. Real Analysis-Modern Techniques and Their Applications, 1984.

Appendix

MAT7067 – Topics in Analysis (2023 Spring)

Lecturer	Bochen LIU
Email	liubc@sustech.edu.cn
Location	Room 309, The 3rd Teaching Building
Time	19:00-21:00, Tuesday in single-week and every Thursday Live Broadcast and video record will be available
Prerequisite	Complex Analysis, Real Analysis, elementary Functional Analysis

1 Course Information

This course consists of two parts.

The first half is the theory of nonharmonic Fourier series. It is concerned with the completeness and expansion properties of sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^2[-\pi, \pi]$. This theory not only has its own interest, but also has many applications in applied math, say in compressed sensing and control theory (we will not get there though).

The second half is Fourier analysis in groups. It gives an exposition of the fundamental ideas and theorems of that Fourier analysis can be developed with minimal assumptions on the nature of the group with which one is working. In particular, it unifies Fourier series and Fourier transform in the representation-theoretic form.

2 Textbook

An Introduction to Nonharmonic Fourier Series (revised 1st edition). Robert M. Young
A course in Abstract Harmonic Analysis (2nd edition). Gerald B. Folland

3 Content

An Introduction to Nonharmonic Fourier Series (revised 1st edition)	
Chapter 1	Bases in Banach Spaces
Chapter 2	Entire Functions of Exponential Type
Chapter 3	The Completeness of Sets of Complex Exponentials
Chapter 4	Interpolation and Bases in Hilbert Space
A course in Abstract Harmonic Analysis (2nd edition)	
Chapter 1	Banach Algebras and Spectral Theory
Chapter 2	Locally Compact Groups
Chapter 3	Basic Representation Theory
Chapter 4	Analysis on Locally Compact Abelian Groups

Assignments