02-14

Main Goal: Generalization of Fourier series on $[-\pi, \pi]$, then

= when certain regularity

$$f(t) \sim \sum_{n\in\mathbb{Z}} a_n e^{int}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

Example of generalization ①: $f \sim \sum_{n=0}^{\infty} a_n e^{i\lambda_n t}$, find $\{\lambda_1, \cdots, \lambda_n, \cdots\}$

由 $n\in\mathbb{Z} \Rightarrow \lambda_n$, for such decomposition

Example of generalization ②: Extend Fourier series to more general ambient space.

e.g. on $[-\pi,\pi]$, $\mathbb{R}^d$, $\mathbb{Z}/p\mathbb{Z}$, etc. $\Rightarrow$ Fourier analysis on group.

We can generalize space mentioned abouve. Here we only consider locally compact Abelian group.

$\Downarrow$

We still have potential to go further.

but that will not be covered in this course (Some French mathematicians)

Reference: Robert Young, and Folland's book.

First 1980 (Revised 2001) originates in Folland's lecture note in 1993.

Other reference: Rudin (Fourier analysis on group) : Hard to read! Assume solid background in Functional Analysis. Folland's book is more self-contained.

1960s

whose book is always harsh to read.

1960s - 1970s (before Stein), when abstract harmonic flourished (by. Rudin)

then Stein (more detailed style), then Bourgain, Wolff, $\cdots$, now.

Trailer: · Fourier inverse theorem $\Longrightarrow$ Pontryagin duality, one of the few example

We can see connection between category theory and analysis.

· Relative topic summer school (2023)

Assessment: · Midterm between Part 1 ~ part 2, only on the first part.

(TBA)

Within 8 weeks, maybe in-class exam, mainly from the reference book

(at most one question from external source)

· Final. (may be on the part 2, TBA)

As is complicated.

Office Hour: Single Tuesday 4-6 p.m. (check e-mail)

Chapter 1: Bases in Banach spaces (only consider infinity-dim space as Finite-space is maily Linear algebra)

Let X be an infinite-dimensional Banach space over $\mathbb{C}$ or $\mathbb{R}$

Def: Hamel basis: maximal linearly independent subset (Existence support by the Zorn's lemma, Axiom of choice), but it's hard to actually find!

Def Schauder basis: $\{x_1, x_2, \cdots\} \subset X$ is a schauder basis for X, if every $x \in X$ corresponds to unique scalars $c_1, c_2, \cdots$, s.t. $x = \sum_{n=1}^{\infty} c_n x_n$, i.e.

$$\lim_{n\to\infty} \| x - \sum_{i=1}^{n} c_i x_i \| = 0$$

Remark: Although Hamel and Schauder basis are different in some way, Schauder basis is the default basis throughout Part 1.

Remark: A Banach space with a basis must be separable

proof: $\{\sum_{\text{finite}} c_i x_i,\ c_i \in \mathbb{Q} + i\mathbb{Q}\}$ $\quad \exists$ a countable dense subset.

e.g. ℓ^∞ has no basis

ℓ^p, $1 \le p < \infty$ has basis $\{(0, \cdots, 1, 0, \cdots)\}$

$\Downarrow$

Banach Asked in 1932: "Does every separable Banach space have a basis?"

Answered by Per Enflo in 1973: No (the counter-example is quite tedious, most familiar examples have basis)

$\Downarrow$

see exercise 1, 3, 6, 7, P2

Section 1.2: Schauder basis for $C[a,b]$

continuous functions on $[a,b]$ with norm $\|f\| = \max_{a\le t\le b} |f(t)|$

Recall the Weierstrass approximation theorem: $\forall \varepsilon > 0$, $\forall f \in C[a,b]$, $\exists$ polynomial P s.t. $\|f - P\| < \varepsilon$

- possible approaches (there are many different ways)

  Bernstein polynomial, $n = 0, 1, 2, \cdots$

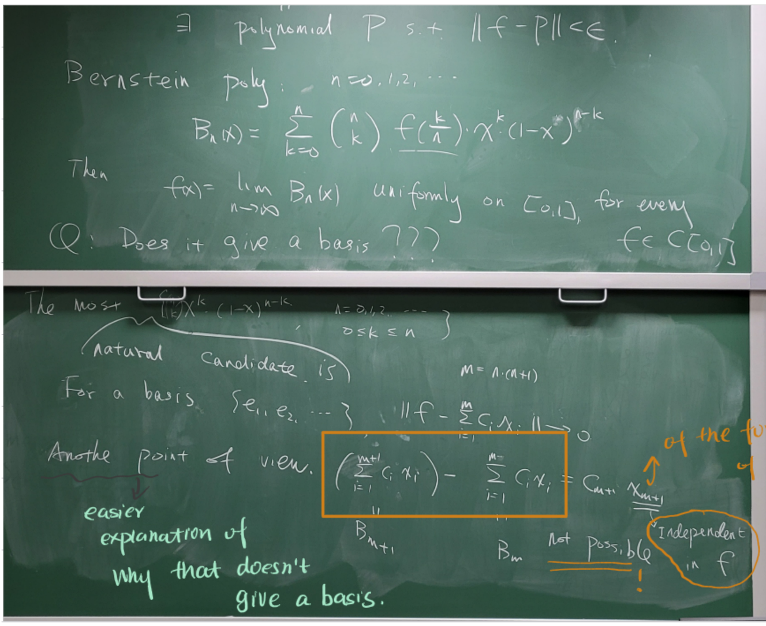
$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

then $f(x) = \lim_{n\to\infty} B_n(x)$ uniformly on $[0,1]$ for every $f \in C[0,1]$

that leads to Q: Does it give a basis? : most natural candidates $\{x^k(1-x)^{n-k},\ n = 0, 1, \cdots,\ 0 \le k \le n\}$

but there will be problem in convergence, as we require $\| f - \sum_{i=1}^{m} c_i \overset{\text{basis}}{x_i} \| \to 0$ (only convergence for a sub-sequence)

Another point of view:

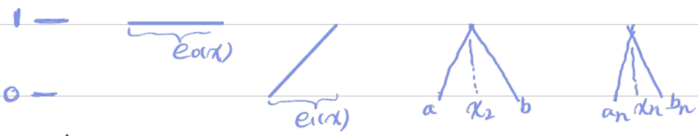


of the form of a basis $\{x^k (1-x)^{n-k}\}$

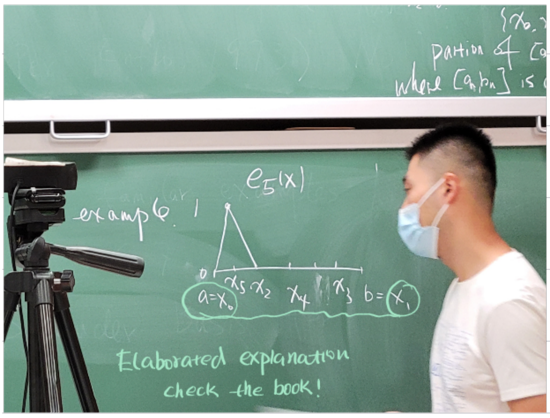
Theorem (Schauder): $C[a,b]$ possess a basis.

Proof: let $\{x_0, x_1, \cdots\} \subset [a,b]$ be a countable dense subset, and $x_0 = a$, $x_1 = b$

and $e_0(x) = 1$, $e_1(x) = \frac{x-a}{b-a}$, $e_2(x)$, $\cdots$, $e_n(x)$

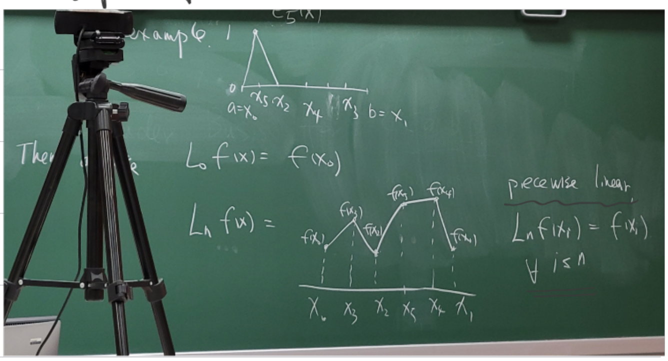


$\{x_0, x_1, \cdots, x_{n-1}\}$ gives a partition of $[a,b]$, and $x_n \in [a_n, b_n]$, where $[a_n, b_n]$ is an interval from this partition



then dense $L_0 f(x) = f(x_0)$

$L_n f(x) =$

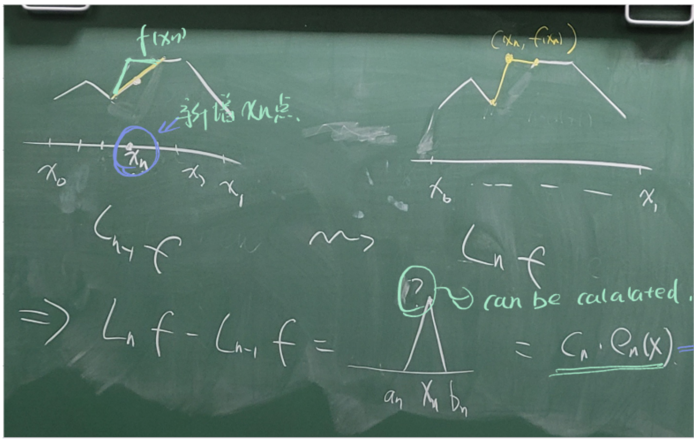


Clearly, $L_n f \to f$ uniformly on $[a,b]$, but how to connect e_n and $L_n(f)$?

First $f = L_0 f + \sum_{n=1}^{\infty} (L_n f - L_{n-1} f)$

$L_0 f = f(x_0) = f(x_0) \cdot e_0(x)$

then what is $L_n f - L_{n-1} f$, see the following illustration



$\Rightarrow$ 可以写成以 $\{e_n\}$ 为基的形式

So what is C_n? In fact $\begin{cases} C_0 = f(x_0) \\ C_n = (f - L_{n-1} f)(x_n) \end{cases}$ $\Rightarrow$ Hence we have shown the convergence now, we may consider the uniqueness.

uniqueness: $f = \sum_{n=0}^{\infty} C_n e_n = \sum_{n=0}^{\infty} C_n' e_n$, then

$0 = \sum_{n=0}^{\infty} (C_n - C_n') e_n$, notice that $e_n(x_i) = 0$, for $i = 0, \cdots, n-1$

let $x = x_0 \Rightarrow C_0 = C_0'$, take $x = x_1 \Rightarrow C_1 = C_1'$

$x = x_2 \Rightarrow C_2 = C_2'$, $\cdots$ $\Rightarrow$ uniqueness.

$\boxed{///}$

Exercise 3.4, 后面我们主要考虑 Hilbert space.

$\Downarrow$

Section 1.3 Orthonormal basis in Hilbert space.

somebook requires Hilbert space to be separable by def

In a separable Hilbert space $\mathcal{H}$. we say $\{e_1, e_2, \cdots\}$ is an othonormal basis, if it is a basis, and $\underbrace{(e_i, e_j)}_{\text{inner product in this Hilbert space.}} = \delta_{ij}$

- An orthonormal basis $\Leftrightarrow$ A complete orthonormal sequence

  $\text{span}\{e_1, \cdots\}^{\perp} = \{0\}$

- Basis expansion $f = \sum \underbrace{(f, e_n)}_{\text{Fourier coefficients}} e_n$

- Parseval's Idenetity $\|f\|^2 = \sum |(f, e_n)|^2$, more generally

  $(f, g) = \sum (f, e_n) \cdot \overline{(g, e_n)}$

$\rightsquigarrow$ Converseny, if $f = \sum_{n=1}^{\infty} (f, e_n) e_n$, where $\{e_n\}$ is an orthonormal sequence then $\{e_n\}$ is a basis (It has the uniqueness)

Proof: It sufficies to show the uniqueness, if

$f = \sum C_n e_n = \sum C_n' e_n \Rightarrow 0 = \sum (C_n - C_n') e_n$, then

$0 = (0, e_n) = (\sum (C_n - C_n') e_n, e_n) = C_n - C_n' \Rightarrow C_n = C_n'$

Note that if we remove the orthogonality, then this result is False!

$f = \sum (f, e_n) e_n \not\Rightarrow$ basis

Example: In $L^2[0,\pi] \subset \underset{\sim\sim\sim}{L^2[-\pi,\pi]}$

we have Fourier series. $f \sim \sum a_n e^{int}$, $a_n = (f, \underbrace{e^{int}}_{e_n})$

For $g \in L^2[0,\pi] \xrightarrow{\text{extend}} \tilde{g} \in L^2[-\pi,\pi]$ (补 0), then $g(t) = \sum (\tilde{g}, e_n)_{L^2(-\pi,\pi)} e_n$ in $[0,\pi]$

$= \sum (g, e_n)_{L^2[0,\pi]} e_n$ in $[0,\pi]$

That leads to $g = \sum (g, e_n) e_n$, $\forall g \in L^2[0,\pi]$, the uniqueness fails, as the extension from $L^2[0,\pi] \to L^2[-\pi,\pi]$ is not unique. (不一定补 0 的 natural extension)

这时 $\{e_n\}$ 在 $[-\pi,\pi]$ 上正交, 但在 $[0,\pi]$ 上积分少了一半, 就不正交了!

Trailer: In $\frac{1}{3}$ of this class, we will consider some scenarios like this (expansion not unique)

$\Downarrow$

Named Fourier frame. Also quite useful.