there is only one section left in this chapter — stability

Section 3.5 : Stability

Recall thm 4: $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is complete if $|\lambda_n| \le n + \frac{1}{2p}$

perturbation of integers

general perturbation

In general, arbitrarily small perturbation does not preserve completeness

e.g. consider $\{e^{i\lambda_n t}\}_{n\neq 0}$ in $L^2[-\pi,\pi]$, $\lambda_n = \begin{cases} n-\frac{1}{4}, & n>0 \\ n+\frac{1}{4}, & n<0 \end{cases}$, we shall show that

this type of examples appears frequently!

① It is complete

② $\forall \varepsilon>0$, $\exists\, \widetilde{\lambda}_n$, $|\widetilde{\lambda}_n - \lambda_n| < \varepsilon$, but $\{e^{i\widetilde{\lambda}_n t}\}$ is $\boxed{\text{not}}$ complete!

① To see the completeness, translate it by $\frac{1}{2}$ to obtain $\{\cdots, -2+\frac{3}{4}, \underset{=\,-1-\frac{1}{4}}{-1+\frac{3}{4}}, 1+\frac{1}{4}, 2+\frac{1}{4}, \cdots\}$, $\underset{=\,0-\frac{1}{4}}{}$

i.e.: n≠0, 少了一个要重新标号

then $|\lambda_n| \le n + \frac{1}{4}$ ✓ by thm 4, completeness.

② : $\forall \varepsilon>0$. let $\widetilde{\lambda}_n = \begin{cases} n-\frac{1}{4}+\varepsilon, & n>0 \\ n+\frac{1}{4}-\varepsilon, & n<0 \end{cases}$, and $\widetilde{\lambda}_0 = 0$ $(n=0)$, then $\{e^{i\widetilde{\lambda}_n t}\} \cup \{e^{i\cdot 0\cdot t}\}$ is a Riesz basis

for $L^2[-\pi,\pi]$, by Kadec's $\frac{1}{4}$ thm. $\Rightarrow \{e^{i\widetilde{\lambda}_n t}\}$ is incomplete 少了 $e^{i\cdot 0\cdot t}$ 一项

additional requirement of perturbation

Thm 11 : If $\lambda_n, \mu_n \in \mathbb{R}$, $\boxed{\sum_n |\lambda_n - \mu_n| < \infty}$, then $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$, $1 \le p < \infty$, then $\{e^{i\mu_n t}\}$ is also complete!

∞ may also be okay!

relies on 换零点:

proof : If not, $\exists\, \phi \neq 0 \in L^{p'}$ s.t. $f(z) = \int_{-\pi}^{\pi} \phi(t)\, e^{izt}\, dt$ vanishes at all μ_n.

Recall that : if $f(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt} dt$, $\phi \in L^{p'}$, $f(\mu) = 0$, then (thm 6)

$$\frac{z-\lambda}{z-\mu} f(z) = \int_{-\pi}^{\pi} \beta(t) e^{izt} dt, \quad \beta \in L^{p'}.$$

Denote $f_0 = f$, $f_n = \frac{z-\lambda_n}{z-\mu_n} f_{n-1}$, by thm 6

$$f_n(z) = \int_{-\pi}^{\pi} \phi_n(t)\, e^{izt}\, dt, \quad \phi_n \in L^{p'}, \text{ and } \boxed{\phi_n(t) = \phi_{n-1} + i(\lambda_n - \mu_n)\, e^{-i\mu_n t} \int_{-\pi}^{t} \phi_{n-1}(s)\, e^{i\mu_n s}\, ds}$$

by Hölder

$$\|\phi_n - \phi_{n-1}\| \le \underbrace{c\cdot|\lambda_n - \mu_n|}_{\text{by def} = \varepsilon_n} \cdot \|\phi_{n-1}\|$$

$$\Rightarrow (1-\varepsilon_n)\|\phi_{n-1}\| \le \|\phi_n\| \le (1+\varepsilon_n)\|\phi_{n-1}\|$$

RHS implies that $\|\phi_n\| \le \underbrace{\prod (1+\varepsilon_n)}_{\text{finite, as } \sum \varepsilon_n < \infty} \|\phi_0\| < \infty$

$$\Rightarrow \|\phi_{n+m} - \phi_n\| \le \sum_{k\ge n} \|\phi_{k+1} - \phi_k\|$$

$$\le \sum_{k \ge n} \varepsilon_k \|\phi_k\|$$

$$\le \underbrace{(\textstyle\sum_{k\ge n} \varepsilon_k)}_{\text{cauchy}} \cdot \underbrace{(\textstyle\prod (1+\varepsilon_n) \|\phi_0\|)}_{\text{finite}} \Rightarrow \{\phi_n\} \text{ cauchy}$$

then $\phi_n \to \widetilde{\phi}$ in $L^{p'}$

indicates that $\{e^{i\lambda_n t}\}$ is not complete!

It remains to show that $\widetilde{\phi} \neq 0$, contradiction to completeness of $\{e^{i\lambda_n t}\}$

$$\|\phi_n\| \ge \underbrace{\textstyle\prod_n (1-\varepsilon_n)}_{>0,\ \text{as } \sum \varepsilon_n < \infty} \|\phi_{n-1}\| > 0$$

∎

End of chapter 3: the last section assume too much background information!

Chapter 4: Interpolation and bases in Hilbert space.

↳ e.g. the Lagrange Interpolation $\sum_i f(x_i) \left(\prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \right)$

In Paley-Wiener space $PW[-\pi, \pi]$, give $\{c_n\} \in \ell^2$, $\exists! \, \phi \in L^2[-\pi,\pi]$, s.t.

$$\int_{-\pi}^{\pi} \phi(t) e^{-int} dt = c_n = f(-n) \text{, Fourier coefficients}$$

given $\{c_n\} \in \ell^2$, $n \in \mathbb{Z}$, then $\exists! \, f \in PW[-\pi,\pi]$ s.t. $f(n) = c_n$

In fact, $\forall f \in PW[-\pi,\pi]$, $f(z) = \sin(\pi z) \cdot \sum_{-\infty}^{\infty} (-1)^n \cdot \frac{f(n)}{\pi(z-n)}$

cardinal sequence

4.1. Moment sequences in $\mathcal{H}$

Def: Given $f_1, f_2, \cdots \in \mathcal{H}$, fixed, we call (f, f_n) the *n-th moment* of f, and $\{(f, f_n)\}$ the moment sequence, and $\{ ((f, f_n))_n : f \in \mathcal{H} \}$ the moment space of $\{f_n\}$

Question: (1) Given $\{f_n\}$, $\{c_n\}$ is $\{c_n\} \in$ moment space ?

i.e. $\exists ? \, f \in \mathcal{H}$ s.t. $(f, f_n) = c_n$?

(2) unique, if exists ? ($\Leftrightarrow$ completeness of $\{f_n\}$)

(3) If not unique, how can the solutions be captured from $\{c_n\}$?

Proposition 1: If a solution exists, then $\exists!$ solution of minimal norm.

proof: unique in span $\{f_n\}$, □ P 123

Example 1. (Finite interpolation) Given linearly independent $\{f_1, f_2, \cdots, f_n\}$, $c_1, \cdots, c_n$, then take

$$f = -\frac{1}{\det(f_i, f_j)} \det \begin{bmatrix} 0 & f_1 & \cdots & f_n \\ c_1 & & & \\ \vdots & & (f_i, f_j) & \\ c_n & & & \end{bmatrix}_{(n+1)\times(n+1)}$$

determined explicitly

notice $\det(f_i, f_j) \neq 0$, as $\langle (a_1, \cdots, a_n), (f_i, f_j) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rangle \geq \|\sum a_i f_i\|^2$

$$(f, f_k) = -\frac{1}{\det(f_i, f_j)} \det \begin{bmatrix} 0, & (f_1, f_k) & \cdots & , (f_n, f_k) \\ c_1 & & & \\ \vdots & & (f_i, f_j) & \\ c_n & & & \end{bmatrix}$$

$$= -\frac{1}{\det(f_i, f_j)} \det \begin{bmatrix} \boxed{-c_k} & 0 \\ c_1 & \\ \vdots & \boxed{(f_i, f_j)} \\ c_n & \end{bmatrix} = c_k$$

Example 2: If $\{f_n\}$ is a Riesz basis, then the moment space is ℓ^2, then $\forall (c_n) \in \ell^2$, one can take

$$f = \sum c_n \underbrace{g_n}_{\text{bi-orthognal sequence}} \in \mathcal{H}$$

then $(f, f_n) = c_n$

Example 3: Say $H = L^2[-\pi,\pi]$, and $\underbrace{\{e^{i\lambda_n t}\}}_{= \{f_n\}}$ is a Riesz basis, then what is $\underbrace{\{g_n\}}_{\text{biorthogonal sequence.}}$?

We shall only consider the special case that $\lambda_{-n} = \lambda_n \in \mathbb{R}$ (in particular $\lambda_0 = 0$)

$\overset{\text{product form of sin, } \frac{\sin\pi z}{\pi z}}{\boxed{\text{In this case } \prod_{n=1}^{\infty}(1-\frac{z^2}{\lambda_n^2}) \in PW[-\pi,\pi]}}$, vanishes at all λ_n but λ_0
assuming this for now, prove later

then $G_1(z) = z\cdot\prod_{n=1}^{\infty}(1-\frac{z^2}{\lambda_n^2})$

$G_1(z)$ vanishes at all λ_n, $G_n = \frac{G_1(z)}{G_1'(\lambda_n)(z-\lambda_n)} \in PW$, vanishes at all λ_j but λ_n.

By Paley-Wiener thm. $\exists\, g_n \in L^2[-\pi,\pi]$, s.t.

$$\int_{-\pi}^{\pi} g_n(t) e^{-i\lambda_m t}\,dt = G_n(\lambda_m) = \delta_{n,m}$$

$\Rightarrow \{g_n\}$ is the bi-orthogonal sequence of $\{e^{i\lambda_n t}\}$

In fact, it is more convenient to translate this discussion to PW.

Then $\{e^{i\lambda_n t}\}$ is a Riesz basis in $L^2[-\pi,\pi]$

$\Updownarrow$

$\{\underbrace{\frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}}_{\text{reproducing kernel}}\}$ is a Riesz basis in $PW[-\pi,\pi]$

Then $\{G_n\}$ is the biorthogonal sequence of $\{\frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}\}$, as $(G_n, \frac{\sin\pi(z-\lambda_m)}{\pi(z-\lambda_m)}) = G_n(\lambda_m) = \delta_{n,m}$

So Given $\{c_n\}\in \ell^2$, $(f, \frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}) = c_n$

$\Longrightarrow$ the solution to this interpolation problem is

$$f(z) = \sum c_n G_n(z) = G_1(z)\sum\frac{c_n}{G_1'(\lambda_n)(z-\lambda_n)} \quad \text{(also unique, by being a basis)}$$

Furthermore, since $(f, \frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}) = c_n \Rightarrow = G_1(z)\sum\frac{f(\lambda_n)}{G_1'(\lambda_n)(z-\lambda_n)}$ $(*)$

$\Updownarrow$

$f(\lambda_n) = c_n$

Also notice $\{f(\lambda_n)\}\in\ell^2$, $\forall f\in PW$, so $(*)$ is valid for all $f\in PW$

$\Downarrow$

a Generalization of Cardinal Series.

$\boxed{\text{It remains to show that } \prod_{n=1}^{\infty}(1-\frac{z^2}{\lambda_n^2}) \in PW[-\pi,\pi]}$ Since $\{e^{i\lambda_n t}\}_{n\neq 0}$ is not complete, $\exists!\, f\in PW$ s.t. (also exponential type)

$f(\lambda_n) = \begin{cases} 0, & n\neq 0 \\ 1, & n=0 \end{cases}$ of deficiency 1.

Claim: f must be a even function

Notice $\tilde{f}(z) = f(-z)$, also satisfies $\tilde{f}(\lambda_n) = \begin{cases} 0, & n\neq 0 \\ 1, & n=0\end{cases}$

By Hadamard Factorization

$$f(z) = e^{Az+B}\underbrace{\prod_{n\neq 0}(1-\frac{z}{\lambda_n})\cdot e^{\frac{z}{\lambda_n}}}_{\text{收敛的}}$$

$= \lim_{N\to\infty}\prod_{n=-N,\, n\neq 0}^{N}(1-\frac{z}{\lambda_n})e^{\frac{z}{\lambda_n}} \underset{\text{$\uparrow$ f even}}{=} \prod_{n\geq 1}(1-\frac{z^2}{\lambda_n^2})$

As $f(0) = 1$, f is even, we have $B=0$, $A=0$.

$\Rightarrow f(z) = \prod_{n\geq 1}(1-\frac{z^2}{\lambda_n^2}) \in PW$ $\boxed{///}$

The following is a general criteria on the existence of a solution

Thm 2: $(f, f_n) = c_n$, admits a solution f with $\|f\|\leq M \iff |\sum a_n\cdot\bar{c}_n| \leq M\cdot\|\sum a_n f_n\|$

$\forall$ finite sequence $\{a_n\}$

proof: "$\Rightarrow$": $|\sum a_n \overline{c_n}| = |\sum a_n \overline{(f, f_n)}|$

$$= |(f, \sum a_n f_n)| \le \|f\| \cdot \|\sum a_n f_n\|$$

$$\le M \cdot \|\sum a_n f_n\|$$

"$\Leftarrow$". consider $T: \sum a_n f_n \to \sum a_n \overline{c_n}$, is a bounded linear functional on $\mathrm{span}\{f_n\}$

正交补部分令为0

of $\|T\| \le M$, It can be further extended to a bounded linear functional on $\mathcal{H}$.

By Riesz representation thm. $\exists f \in H$. $\|f\| \le M$, s.t. $(\sum a_n f_n, f) = \sum a_n \overline{c_n}$

$$\Rightarrow (f, f_n) = c_n$$

$\boxed{///}$

Exercise 5.