We will finish this book within $\boxed{\text{2 classes}}$

Some notation (about Riesz basis) are still quite common in recent research

despite difference in setting

less relies on complex analysis

In higher-dim. complex analysis is not that useful. (零点不孤立)

4.2 Bessel sequences and Riesz-Fischer sequences $(\{f_n\} \subset H)$

Def: (Bessel) $\sum |\langle f, f_n\rangle|^2 < \infty$, $\forall f \in H$

(R-F) $\forall (c_n) \in \ell^2$, $\exists f \in H$, s.t. $\langle f, f_n\rangle = c_n$

Equivalently: moment space of Bessel $\subset \ell^2 \subset$ moment space of R-F

Remark: "=" Riesz sequence = Bessel + R-F

Riesz sequence + completeness = Riesz basis.

Proposition 2: Bessel $\Leftrightarrow \underbrace{\sum |\langle f, f_n\rangle|^2 \le M \|f\|^2}_{\text{by Banach-Steinhaus thm.}}$ 类似上界

Riesz-Fischer $\Leftrightarrow \exists m > 0$, s.t. $\forall (c_n) \in \ell^2$, $\exists f \in H$, s.t. $\underbrace{\|f\|^2 \le \frac{1}{m} \sum |c_n|^2}_{\text{uniformly bounded solution}}$ 类似下界

actually an exercise in the book.

proof: consider $\ell^2 \xrightarrow{T} H / \operatorname{span}\{f_n\}^\perp$ (mod) : well-defined, linear

we shall show that T is bounded.

Say $\alpha_k \to \alpha \in \ell^2$, $\alpha_k = (c_{nk})_n$, $\alpha = (c_n)_n$.

$T\alpha_k \to \beta \in H / \operatorname{span}\{f_n\}^\perp$

$\langle \beta, f_n\rangle = \lim_{k\to\infty} \langle T\alpha_k, f_n\rangle = \lim_{k\to\infty} (c_{nk}) = c_n$

} by the closed graph thm, T is bounded. ∎

by def $T\alpha = \beta$

Thm 3: (1) $\{f_n\}$ is Bessel with bound M

$\Leftrightarrow \|\sum c_n f_n\| \le M \cdot \sum |c_n|^2$, $\forall$ finite sequence $\{c_n\}$

(2) R-F with bound $m \Leftrightarrow$

$m \sum |c_n|^2 \le \|\sum c_n f_n\|^2$, $\forall$ finite sequence $\{c_n\}$.

proof: If and only if

(1) Bessel : $T: f \mapsto (\langle f, f_n\rangle)$ is bounded by M.

$\langle T^*(b_n)_n, f\rangle = \sum b_n \overline{\langle f, f_n\rangle} = \langle \sum b_n f_n, f\rangle$

then it follows from $\|T\| = \|T^*\|$

(2) "$\Rightarrow$" let f be a solution of $\langle f, f_n\rangle = c_n$.

$\|f\|^2 \le \frac{1}{m}\sum |c_n|^2$, then $m\cdot\sum|c_n|^2 = m\cdot\sum_i c_n\cdot\overline{\langle f, f_n\rangle}$

$$= m\cdot\langle\sum c_n f_n, f\rangle$$

$$\le m\|\sum c_n f_n\|\cdot\|f\|$$

$$\overset{\text{prop 2}}{\le} \sqrt{m}\,(\sum_i|c_n|^2)^{\frac{1}{2}}\cdot\|\sum c_n f_n\|$$

then done.

"$\Leftarrow$" Recall (last lecture) thm 2. $\langle f, f_n\rangle = c_n$ has solution of norm $\le M$ if $|\sum a_n\overline{c_n}| \le M\cdot\|\sum a_n f_n\|$ $\forall$ finite sequence.

To check this condition, by Cauchy-Schwarz

$$|\sum a_n\overline{c_n}|^2 \le \sum|a_n|^2\cdot\sum|c_n|^2$$

$$\le \frac{1}{m}\underbrace{\sum|c_n|^2}_{<\infty,\ \text{as } (c_n)\in\ell^2}\cdot\|\sum a_n f_n\|^2$$

by thm 2 $\Rightarrow$ $\exists$ a solution f s.t. $\|f\|^2 \le \frac{1}{m}\cdot\sum|c_n|^2$ $\boxed{//}$

Remark: (In operator language,) Bessel of bound $M \Leftrightarrow T: e_n \to f_n$. $\|T\| \le \sqrt{M}$

$$(\ \|\sum c_n f_n\|^2 \le M\cdot\underbrace{\sum|c_n|^2}_{=\|\sum c_n e_n\|^2}\ )$$

R-F of bound $m \Leftrightarrow S: f_n \to e_n$, $\|S\| \le \sqrt{\frac{1}{m}}$

In the language of Gram-matrix

$\langle f_i, f_j\rangle_{ij} \overset{\text{def}}{=} A$, then

Bessel $\Leftrightarrow$ $\|A\| \le M$ on ℓ^2

R-F $\Leftrightarrow$ every $n\times n$ sub-matrix A_n of A satisfies $m\|c\|^2 \le \|A_n c\|$ $\forall\, c = (c_1, \cdots, c_n)$

Example: e.g. $\{1, t, t^2, \cdots\}$ is Bessel in $L^2[0,1]$, whose gram matrix $(\frac{1}{i+j+1})_{ij}$ that has norm π on ℓ^2, but not Riesz-Fischer, $\|f_n\| \geq c > 0$, while $\|t^n\| \to 0$.

Thm 4: If $\lambda_n \in \mathbb{R}$, separated ($|\lambda_n - \lambda_m| > \delta > 0$, $\forall n \ne m$), then $\{e^{i\lambda_n t}\}$ is Bessel sequence in $L^2[-A, A]$, $\forall\, 0 < A < \infty$.

proof: $f \in PW$, then $f(z) = \int_{-A}^{A}\phi(t)e^{izt}dt$

$$\underset{\substack{\| \\ \sum|\langle\phi, e^{i\lambda_n t}\rangle|^2}}{\sum|f(\lambda_n)|^2} \le \underset{\substack{\| \\ c\cdot\|\phi\|^2_{L^2}}}{c\int_{-\infty}^{+\infty}|f(x)|^2}$$

then we know it's a Bessel sequence. $\boxed{//}$

Section 4: The moment space and equivalent sequences.

Def: $\{f_n\}$, and $\{g_n\}$ are called equivalent if $\exists$ T bounded invertible $Tf_n = g_n$

only 1 thm in this section

Thm 7. Two complete sequences are equivalent if and only if they have the same moment space.

Cor: Completeness + Riesz sequence = Riesz basis

ℓ^2 is moment space $\{e_n\}$

moment space ℓ^2

Proof of thm 7: "$\Rightarrow$" $Tf_n = g_n$, Given $\langle f, f_n \rangle$, we need to find g s.t. $\langle g, g_n \rangle = \langle f, f_n \rangle$, $\forall n \in \mathbb{N}$

Since T is invertible, only need to prove

$$\langle f, f_n \rangle = \langle f, T^{-1} g_n \rangle = \langle \underbrace{(T^{-1})^* f}_{g}, g_n \rangle$$

then moment space of $\{f_n\}$ $\subseteq$ moment space of $\{g_n\}$

the other direction is similar.

"$\Leftarrow$" $\langle f, f_n \rangle = \langle g, g_n \rangle$ defines a bijection $f \leftrightarrow g$, and linear

We still need to show that it's bounded!

define $Tf = g$, use the $\boxed{\text{closed-graph thm}}$

We shall show that T is bounded, say $f_k \to f$, $Tf_k \to g$, then

$$\langle g, g_n \rangle = \lim_{k \to \infty} \langle Tf_k, g_n \rangle = \lim_{k \to \infty} \langle f_k, f_n \rangle = \langle f, f_n \rangle$$

$\Rightarrow$ $Tf = g$ as desired.

The other direction is similar $\Rightarrow$ T is invertible. Finally

$$\langle f, f_n \rangle = \langle Tf, g_n \rangle = \langle f, T^* g_n \rangle \Rightarrow T^* g_n = f_n$$ ///

Now, we will discuss stability of Riesz basis

still a popular topic in recent study.

Section 6: Interpolation in PW · stability

Def: $\{\lambda_1, \lambda_2, \cdots\} \subseteq \mathbb{C}$ is called an interpolating sequence, if

$$\{ (f(\lambda_n))_n : f \in PW \} = \ell^2$$

$$\| $$

$$\Updownarrow \quad \{ (\int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t} dt)_n : \phi \in L^2[-\pi, \pi] \}$$

the moment space of $\{e^{-i\lambda_n t}\}$ is ℓ^2 $\Leftrightarrow$ $\{e^{-i\lambda_n t}\}$ is a Riesz sequence for $L^2[-\pi, \pi]$

If in addition, the solution for $f(\lambda_n) = c_n$ is unique, we call $\{\lambda_n\}$ complete interpolating sequence

$\Leftrightarrow$ $\{e^{-i\lambda_n t}\}$ is a Riesz basis for $L^2[-\pi, \pi]$.

Proposition: If $\{\lambda_1, \lambda_2, \cdots\} \subset \mathbb{C}$ is an interpolating sequence, then it must lie in a horizontal strip (对Im有要求), and be separated.

Proof: We first show it lies in a horizontal strip. Since it is Bessel, $\|\sum c_n f_n\|^2 \le M \cdot \sum |c_n|^2$

$$\Rightarrow \|f_n\|^2 \le M, \text{ uniformly in } n$$

$$\|f_n\|^2 = \int_{-\pi}^{\pi} e^{2 \cdot \mathrm{Im} z t} dt \sim \frac{e^{2\pi \cdot |\mathrm{Im} z|}}{|\mathrm{Im} z|} \quad \text{bounded only if } |\mathrm{Im} z| \text{ is bounded.}$$

Say $|\mathrm{Im} z| \le H$.

Then we prove it is separated. Since $\{e^{i\lambda_n t}\}$ is R-F, by prop-2, $\forall (c_n) \in \ell^2$, $\exists f$ s.t. $(f, e^{i\lambda_n t}) = c_n$, $\|f\|^2 \le \frac{1}{m} \sum |c_n|^2$

$\Rightarrow \forall m, \exists f_k$ s.t. $(f_k, e^{i\lambda_n t}) = \delta_{n,k}$, $\|f_k\| \le \frac{1}{m}$

Denote $F_k(z) = \int_{-\pi}^{\pi} f_k(t) e^{-izt} dt$, then $F_k(\lambda_n) = \delta_{n,k}$.

$$1 = |F_k(\lambda_n) - F_k(\lambda_k)| = \left|\int_{\lambda_k}^{\lambda_n} F_k'(z) dt\right|$$

$$\le |\lambda_n - \lambda_k| \cdot \sup_{|\mathrm{Im} z| \le H} |F_k'|$$

Notice that $\sup |F_k'(z)| = \int_{-\pi}^{\pi} it f_k(t) e^{izt} dt$

$$\le \pi e^{H\pi} \|f_k\|_{L^2} \le \pi \cdot e^{H\pi} \cdot \frac{1}{m}$$

$\Rightarrow |\lambda_n - \lambda_k| \ge (\pi \cdot e^{H\pi} \cdot \frac{1}{m})^{-1} > 0 \Rightarrow$ separated! $\boxed{///}$

Our goal: If $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2[-\pi, \pi]$, then $\exists L > 0$, s.t. $\{e^{i\mu_n t}\}$ is also a Riesz basis if $\underbrace{|\mu_n - \lambda_n| < L}_{\text{later, we first see section 7}}$ (recall it fails for completeness!)

Section 7: The theory of frame

Def: $\{f_n\} \subset H$ is called a frame, if $\exists A, B > 0$, s.t.

$$A\|f\|^2 \le \sum |(f, f_n)|^2 \le B\|f\|^2$$

(Riesz sequence: $A \sum |c_n|^2 \le \|\sum c_n f_n\|^2 \le B \sum |c_n|^2$)

Remark ① It's Bessel (by RHS) $\overset{\text{thm 3}}{\Longleftrightarrow} \|\sum c_n f_n\|^2 \le B \cdot \sum |c_n|^2$

② It must be complete. (by LHS)

③ union of frame is also a frame (not a good property, frame 不一定为 separated, 尽管有 stability!)

Example: ①: Every orthonormal basis is a frame

②: $\{e^{int}\}$ is a frame for $L^2[-A, A]$, $\forall A \le \pi$ ($A < \pi$ 时不一定为 basis.)

Recall that $L^2[-A, A] \xrightarrow{\text{extend}} L^2[-\pi, \pi]$ 再找 Fourier 系数再限制回来
但 extension 不唯一
$\Downarrow$
不一定为 basis

More generally, frame for H is a frame for every subspace H', may not be a basis!

③: In PW, it means $A\int_{-\infty}^{\infty}|f(x)|^2dx \le \sum_n |f(\lambda_n)|^2 \le B\int_{-\infty}^{\infty}|f(x)|^2dx$

Now, give a frame $\{f_n\}$. consider $Tf = \sum \langle f, f_n\rangle f_n$

It's bounded as $\{f_n\}$ is Bessel $\Rightarrow \|\sum\langle f,f_n\rangle f_n\|^2 \le \sum|\langle f,f_n\rangle|^2 \le B\cdot\|f\|^2$

We shall show that T is invertible.

Notice $\underbrace{\langle Tf,f\rangle}_{\le\|Tf\|\|f\|} = \langle\sum\langle f,f_n\rangle f_n, f\rangle = \sum|\langle f,f_n\rangle|^2 \ge A\|f\|^2$

$\|Tf\|\|f\| \ge$ cauchy $\Rightarrow \|Tf\| \ge A\|f\|$

Also notice that T is self-adjoint

$\langle Tf, g\rangle = \sum_i \langle f,f_n\rangle\overline{\langle g,g_n\rangle} = \langle f, Tg\rangle$

If T is not onto, $\exists g \in \mathrm{range}(T)^\perp\setminus\{0\}$

$\Rightarrow 0 = \langle T(Tg), g\rangle = \|Tg\|^2 \ge A^2\|g\|^2 > 0$, contradiction.

Hence T is onto, then by $\boxed{\text{open mapping theorem}}$

T is invertible and therefore $f = \sum\langle T^{-1}f, f_n\rangle f_n$

Lemma 5: Given a frame, $f = \sum a_n f_n$ is unique if we require $a_n = \langle g, f_n\rangle$, for some $g\in H$

Moreover, if $f = \sum b_n f_n$ for some other (b_n), then

$$\sum_i |b_n|^2 = \sum|a_n|^2 + \sum_i|b_n - a_n|^2 \quad (\ge \sum|a_n|^2)$$

Remark: coefficients given by $a_n = \langle g, f_n\rangle$ is "minimal"

proof: Existence of $g\in H$ ✓

uniqueness: say $f = \sum\langle h, f_n\rangle f_n = \sum_i\langle T^{-1}(Th), f_n\rangle f_n$

$= Th$

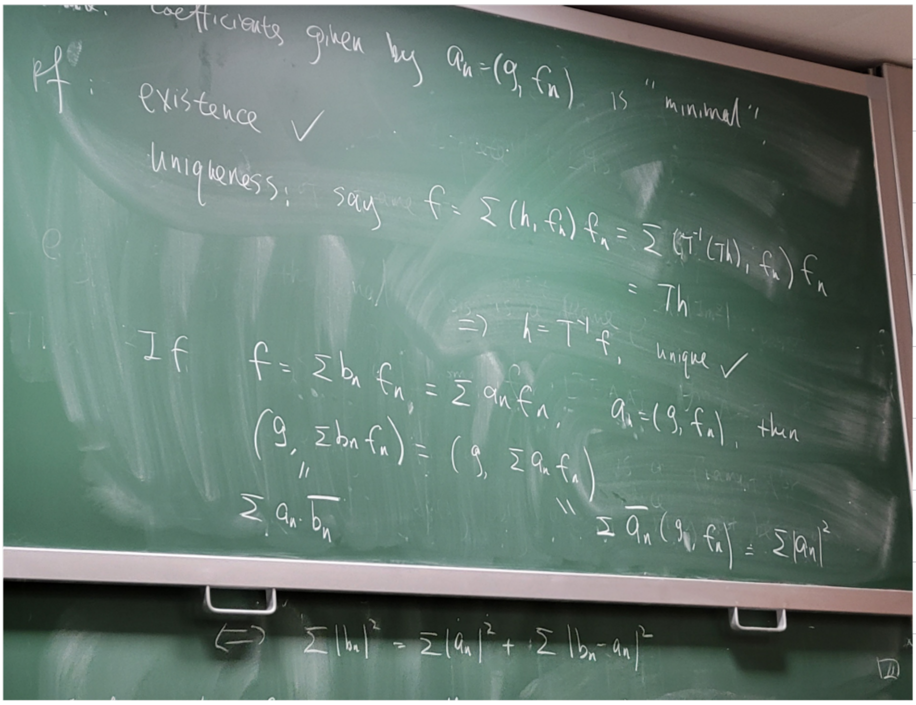
$\Rightarrow h = T^{-1}f$, unique

If $f = \sum b_n f_n = \sum a_n f_n$, $a_n = \langle g, f_n\rangle$, then

$\langle g, \sum b_n f_n\rangle = \langle g, \sum a_n f_n\rangle$

$\sum a_n \overline{b_n} \qquad \sum \overline{a_n}\langle g, f_n\rangle = \sum|a_n|^2$

$\Leftrightarrow \sum|b_n|^2 = \sum_i|a_n|^2 + \sum|b_n - a_n|^2$ ◻



Def: A frame is called exact if it fails to be a frame when any term is removed

We shall prove Riesz basis = exact frame

thus = complete Riesz sequence.

also = frame + Riesz sequence.

Stability.

小测 (再下周)

简单 thm 应用 (all required thms, defs will be listed!)