Recap: Bessel sequence: $\sum |\langle f, f_n\rangle|^2 < \infty$, $\forall f \in H$, $\{f_n\} \subset H$

$\Leftrightarrow \sum |\langle f, f_n\rangle|^2 \le M \|f\|^2$ (Banach–Steinhaus)

$\Leftrightarrow \|\sum c_n f_n\|^2 \le M \cdot \sum |c_n|^2$, $\forall$ finite sequence $\{c_n\}$

Riesz-Fischer: $\forall (c_n) \in \ell^2$, $\exists$ a solution $f \in H$ to the equations $\langle f, f_n\rangle = c_n$

$\Leftrightarrow \exists m$, s.t. $\exists$ a solution $\|f\|^2 \le \frac{1}{m} \sum |c_n|^2$, $\forall (c_n) \in \ell^2$
uniformly.

f is unique with min-norm.

$\Leftrightarrow m \sum |c_n|^2 \le \|\sum c_n f_n\|^2$, $\forall$ finite sequence (c_n)

Def: Riesz sequence = Bessel + Riesz-Fischer, i.e.

$$m \sum |c_n|^2 \le \|\sum c_n f_n\|^2 \le M \cdot \sum |c_n|^2$$

$\Leftrightarrow$ moment space $\{(\langle f, f_n\rangle), f \in H\} = \ell^2$

$\{e^{i\lambda_n t}\}$ is a Riesz sequence for $L^2[-\pi, \pi]$

$\stackrel{\text{def}}{\Leftrightarrow}$ $\{\lambda_n\}$ is an interpolating sequence for PW

$\Big\}$ + completeness = Riesz basis.

Frame: $\{f_n\}$ $A\|f\|^2 \le \underbrace{\sum |\langle f, f_n\rangle|^2 \le B\|f\|^2}_{\text{Bessel.}}$, complete, but not necessarily a basis.

Given a frame, $f = \sum \langle T^{-1}f, f_n\rangle f_n$, T invertible.

$\exists !$ representation s.t. $f = \sum \langle g, f_n\rangle f_n$,

$\forall$ representation $f = \sum a_n f_n$

$$\sum |\langle T^{-1}f, f_n\rangle|^2 \le \sum |a_n|^2$$

Def (Exact frame): We shall show that exact frame = Riesz basis ← last lecture

Lemma 6: the removal of a vector from a frame leaves either a frame or an incomplete set.

Proof: say we remove f_m, since $\{f_n\}$ is a frame, $\exists !$ $f_m = \sum_n \langle g_m, f_n\rangle f_n$ ← see above recap

case ①: $\langle g_m, f_m\rangle = 1$, by the "minimality" of $\sum \langle g_m, f_n\rangle f_n$,

$\sum |\langle g_m, f_n\rangle|^2 \le 1 \longrightarrow f_m = \sum \delta_{n,m} f_n$
$\|$
$1 + \sum_{m\ne n} |\langle g_m, f_n\rangle|^2 \Rightarrow \langle g_m, f_n\rangle = \delta_{n,m} \Rightarrow \{f_n\}_{n\ne m}$ is incomplete.

是f_n的线性组合

case ②: $\langle g_m, f_m\rangle \ne 1 \Rightarrow f_m = \sum_{n\ne m} b_n f_n$, $0 < \sum |b_n|^2 < \infty$

then $\forall f \in H$, $|\langle f, f_m\rangle|^2 = |\sum_{n\ne m} b_n \langle f, f_n\rangle|^2 \le \sum |b_n|^2 \cdot \sum_{n\ne m} |\langle f, f_n\rangle|^2$.

Now we show that $\{f_n\}_{n\neq m}$ is a frame. Upper bound is trivial.

$$\Downarrow$$

$$\sum_{n\neq m} |\langle f, f_n\rangle|^2 \le \sum_n |\langle f, f_n\rangle|^2 \le B\|f\|^2$$

For lower bound.

by above. in ②

$$A\|f\|^2 \le \sum_n |\langle f, f_n\rangle|^2 = \sum_{n\neq m} \sim + |\langle f, f_m\rangle|^2 \le \boxed{\left[1+\sum |b_n|^2 M\right]} \cdot \sum_{n\neq m} |\langle f, f_n\rangle|^2$$

$\Rightarrow$ $\{f_n\}_{n\neq m}$ is a frame.

Remark: Give an exact frame, then $\{f_n\}$, then $\{f_n\}_{n\neq m}$ is not a frame case ② 不成立

$\Rightarrow$ $\langle g_m, f_n\rangle = \delta_{n,m}$ by the proof above.

$\Rightarrow$ Every exact frame admits a bi-orthogonal sequence. useful in inner-product representation

Thm 12: Exact frame = Riesz basis

"$\Leftarrow$": $\exists$ invertible T, s.t. $Te_n = f_n$, so

$$\sum |\langle f, f_n\rangle|^2 = \sum |\langle T^* f, e_n\rangle|^2 = \|T^* f\|^2 \approx \|f\|^2$$

$\Rightarrow$ $\{f_n\}$ is a frame.

As $\{f_n\}_{n\neq m}$ must be incomplete, thus not a frame, Overall Riesz basis is an exact frame.

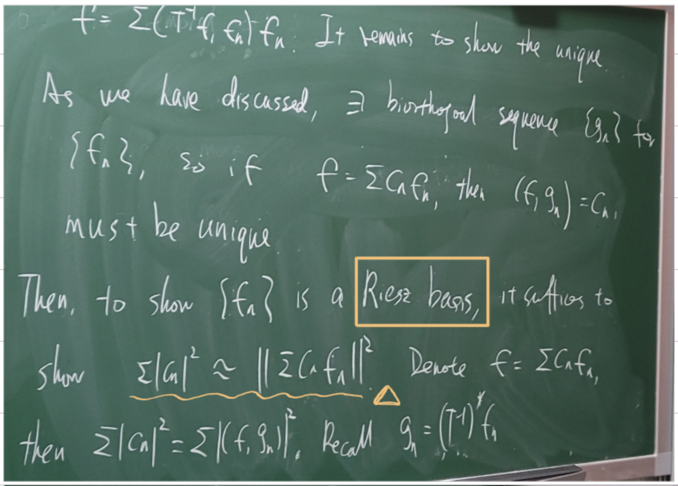
"$\Rightarrow$" We first show that it's a basis, since it's a frame, $\boxed{f = \sum \langle T^{-1} f, f_n\rangle f_n}$ check recap.

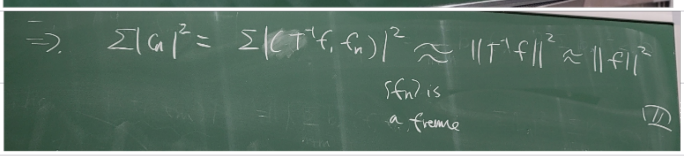
It remains to show that it's unique.

As we have discussed, $\exists$ biorthogonal sequence $\{g_n\}$ for $\{f_n\}$

So if $f = \sum c_n f_n$, then $\langle f, g_n\rangle = c_n$, must be unique.

Then to show that $\{f_n\}$ a Riesz basis, it sufficies to show





Section 8: $\boxed{\text{Stability}}$ of Non-harmonic Series.

Recall $\underbrace{\text{Riesz basis} = \text{Riesz sequence} + \text{frame}}_{\text{both Bessel}}$.

we first show a result of Bessel sequence.

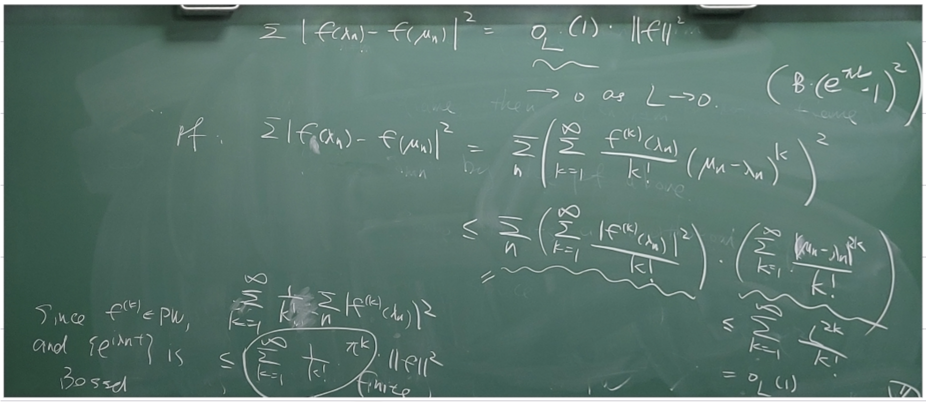
a simpler version, compared to the lemma in textbook.

Lemma 3: If $f \in PW$, $\sum |f(\lambda_n)|^2 \le B \cdot \|f\|^2$, then $\forall \{\mu_1, \cdots\}$, $\sup_n |\lambda_n - \mu_n| \le L < \infty$, then

$$\sum |f(\lambda_n) - f(\mu_n)|^2 = \underbrace{o_L(1)}_{\to 0 \text{ as } L \to 0} \cdot \|f\|^2$$

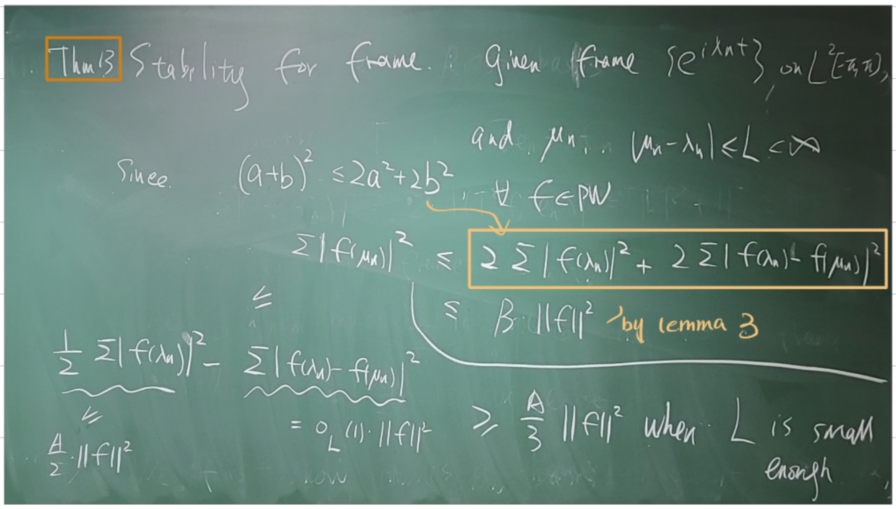
proof: $\sum_n |f(\lambda_n) - f(\mu_n)|^2 = \sum_n \left(\sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{k!} (\mu_n - \lambda_n)^k\right)^2$

$$\le \sum_n \left(\sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^2}{k!}\right) \cdot \underbrace{\left(\sum_{k=1}^{\infty} \frac{|\mu_n - \lambda_n|^{2k}}{k!}\right)}_{\le \sum_{k=1}^{\infty} \frac{L^{2k}}{k!} = O_L(1)}$$



Now, the $\boxed{\text{stability}}$ of frame: Thm B Given frame $\{e^{i\lambda_n t}\}$ on $L^2[-\pi, \pi]$, and μ_n, $|\mu_n - \lambda_n| \le L < \infty$

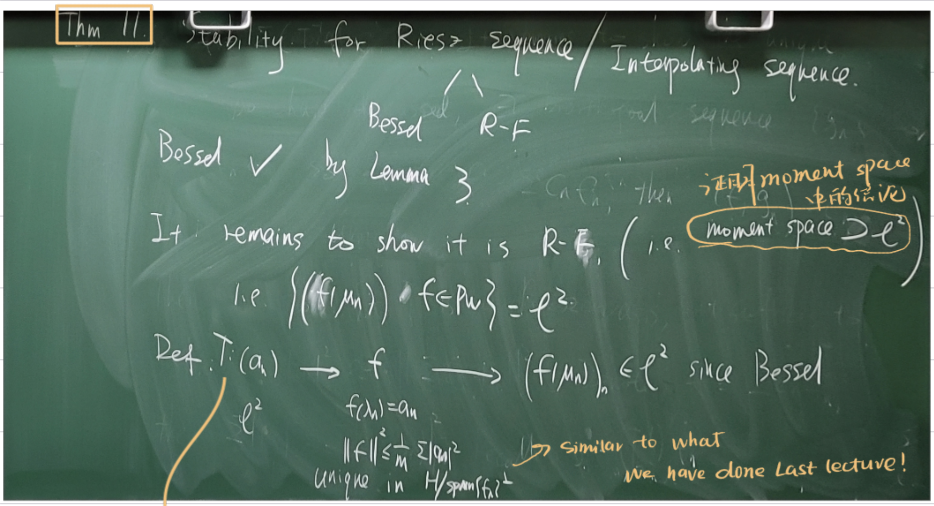
Since $(a+b)^2 \le 2a^2 + 2b^2$, $\forall f \in PW$



Now for stability of Riesz sequence / Interpolating sequence. (Thm II)

$\overbrace{\text{Bessel} \quad \text{R-F}}$

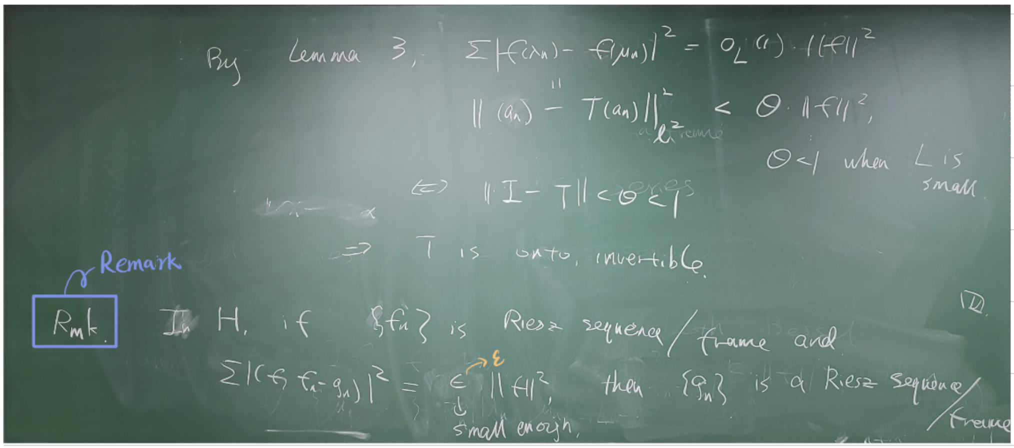
Bessel: Done by lemma 3.



By lemma 3. $\sum |f(\lambda_n)-f(\mu_n)|^2 = O_L(1)\cdot\|f\|^2$

$$\|(a_n)-T(a_n)\|^2_{\ell^2} < \theta\cdot\|f\|^2 \quad . \ \theta<1. \text{ when } L \text{ is small.}$$

$\iff \|I-T\|<\theta<1 \Rightarrow T$ is onto, invertible! (see previous lecture!)



by stability, we may assume $\{\lambda_n\}$ as rational numbers!

Now. we show the last theorem in this book.

4.9 : Pointwise – convergence.

Def: $\sum a_n$, $\sum b_n$ are said to be $\boxed{\text{equi-convergent}}$ if $\sum_{n=1}^{N}(a_n-b_n)\to 0$

proof is a little complicated.

Thm 15: If $\{e^{i\lambda_n t}\}$ is a $\boxed{\text{Riesz basis}}$ for $L^2[-\pi,\pi]$, and $\sup_n |\lambda_n - n| \overset{\text{def}}{=} L<\infty$, then $\forall f\in L^2[-\pi,\pi]$.

the ordinary Fourier Series and non-harmonic Fourier series are uniformly equi-convergent

on every cpt subset of $(-\pi,\pi)$.

proof: $f=\sum a_n e^{int} = \sum c_n e^{i\lambda_n t}$, we need to estimate the partial sum

$$\sum_{-N}^{N}(a_n e^{int} - c_n e^{i\lambda_n t})$$

To do this, we shall find a good representation.

$$\text{write } e^{i\lambda_n t} = e^{int}\cdot e^{i(\lambda_n - n)t}$$

Pf. $f = \sum a_n e^{int} = \sum c_n e^{i\lambda_n t}$. We need to estimate the partial sum $\sum_{-N}^{N} (a_n e^{int} - c_n e^{i\lambda_n t})$. To do this we shall find a good representation.

Write $e^{i\lambda_n t} = e^{int} \cdot e^{i(\lambda_n - n)t}$

$$= e^{int} \boxed{\sum_{k=0}^{\infty} \frac{(i(\lambda_n - n)t)^k}{k!}}$$

$$:= e^{int} \sum_{k=0}^{\infty} b_{nk} t^k, \text{ then}$$

Then $f_N := \sum_{-N}^{N} c_n e^{i\lambda_n t}$

$$= \sum_{-N}^{N} c_n e^{int} \cdot \sum_{k=0}^{\infty} b_{nk} t^k$$

$$= \sum_{k=0}^{\infty} \underbrace{\left(\sum_{n=-N}^{N} c_n b_{nk} e^{int} \right)}_{:= \psi_{Nk}} t^k \quad \text{def} = \psi_{N,k}$$

We shall show $f = \sum_{k=0}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n b_{nk} e^{int} \right) t^k$ → Goal. To see this,

① We first show that $\lim_{N\to\infty} \psi_{Nk} = \psi_k \overset{\text{def}}{:=} \sum_{n=-\infty}^{\infty} c_n b_{nk} e^{int}$, in L^2

内层

this is because.

This is because $(c_n) \in \ell^2$, and $|b_{nk}| \le \frac{L^k}{k!} < \infty$ uniformly in k.

$\Rightarrow (c_n b_{nk})_n \in \ell^2$, so $\psi_{Nk} \to \psi_k$ in L^2

② Second, we show $\sum \psi_k t^k \to f$ is $\boxed{\text{Cauchy}}$ (cauchy) in L^2

外层

Notice $\|\psi_k\|_{L^2}^2 = \sum_n |c_n|^2 \cdot |b_{nk}|^2 \le \frac{L^{2k}}{(k!)^2} \sum |c_n|^2$

$$\Rightarrow \left\| \sum_{k>M} \psi_k t^k \right\| \le \sum_{k>M} |\pi|^k \cdot \|\psi_k\| \le \sum_{k>M} \frac{(\pi L)^k}{k!} \cdot \left(\sum_n |c_n|^2\right)^{\frac{1}{2}}$$

取范数 TV

then cauchy.

③ $f = \sum \psi_k t^k$. To see this, consider

$$\left\| \sum \psi_k t^k - f_N \right\| = \left\| \sum_{k=0}^{\infty} \psi_k t^k - \sum_{k=0}^{\infty} \sum_{k=-N}^{N} c_n b_{nk} e^{int} t^k \right\|$$

$$= \left\| \sum_{k=0}^{\infty} \left(\sum_{|k|>N} c_n b_{nk} e^{int} \right) t^k \right\|$$

取范数 与②类似

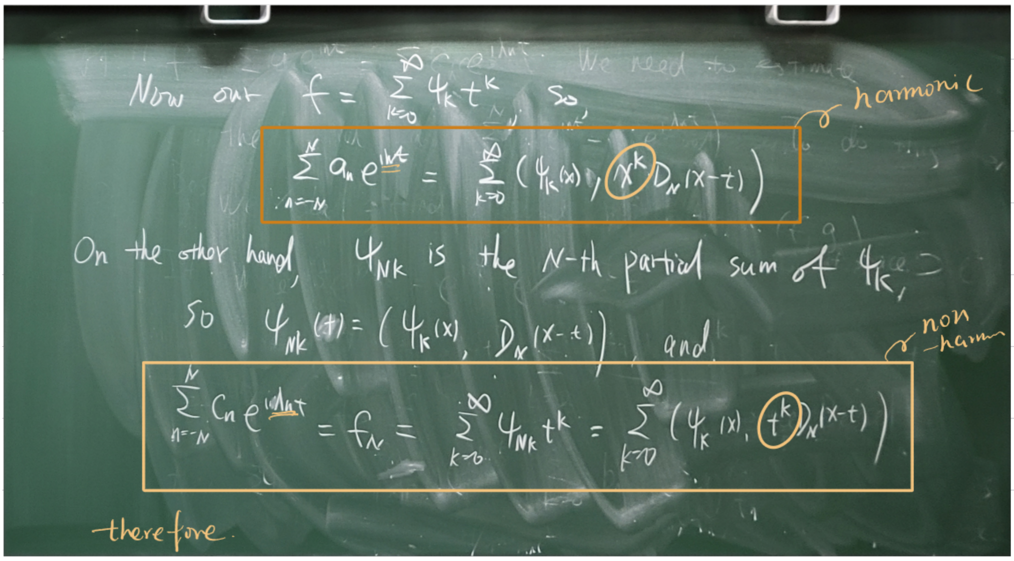
$$\le \sum_{k=0}^{\infty} \frac{(\pi L)^k}{k!} \cdot \sum_{n>|N|} |c_n| \to 0$$

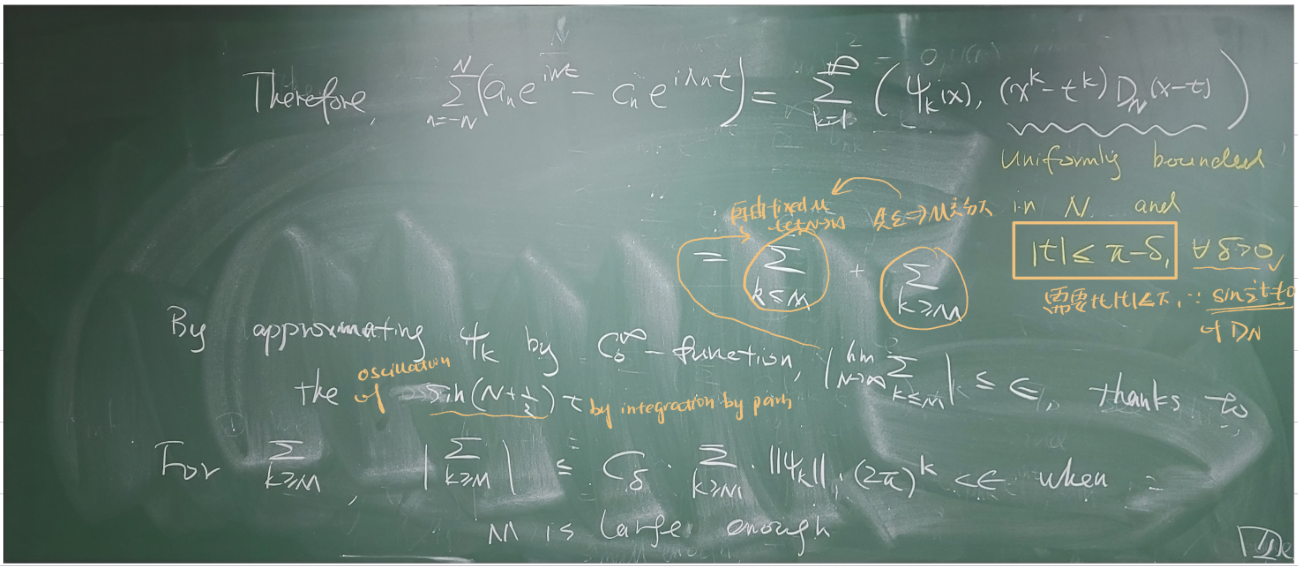
Then, recall the $\boxed{\text{Dirichlet kernel}}$ → 部分和的内积表达 c.f. stein-complex

$D_N(t) = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}$; say

$f = \sum a_n e^{int}$, then $\sum_{-N}^{N} a_n e^{int} = (f(x), D_N(x-t))$ → 积分

求和 积分 → 交换 先求和 再积分


Now our $f = \sum_{k=0}^{\infty} \varphi_k t^k$ so,
harmonic
$\sum_{n=-N}^{N} a_n e^{int} = \sum_{k=0}^{\infty} \left(\varphi_k(x), x^k D_N(x-t) \right)$
On the other hand, φ_{Nk} is the N-th partial sum of φ_k,
so $\varphi_{Nk}(t) = \left(\varphi_k(x), D_N(x-t) \right)$ and
non-harmonic
$\sum_{n=-N}^{N} c_n e^{i\lambda_n t} = f_N = \sum_{k=0}^{\infty} \varphi_{Nk} t^k = \sum_{k=0}^{\infty} \left(\varphi_k(x), t^k D_N(x-t) \right)$
therefore.


Therefore, $\sum_{n=-N}^{N} \left(a_n e^{int} - c_n e^{i\lambda_n t} \right) = \sum_{k=1}^{\infty} \left(\varphi_k(x), (x^k - t^k) D_N(x-t) \right)$
Uniformly bounded in N and $|t| \le \pi - \delta$, $\forall \delta > 0$
of D_N
再由 fixed M, let $N \to \infty$
$= \sum_{k \le M} + \sum_{k > M}$
By approximating φ_k by C_0^{∞}-function, $\left| \lim_{N\to\infty} \sum_{k \le M} \right| \le \epsilon$, thanks to
the oscillation of $\sin(N+\frac{1}{2})t$ by integration by parts
For $\sum_{k \ge M}$, $\left| \sum_{k \ge M} \right| \le C_\delta \cdot \sum_{k \ge M} \|\varphi_k\|_1 \cdot (2\pi)^k < \epsilon$ when
M is large enough

下周2，3个题