Ref: "A course in Abstract Harmonic Analysis"

Chapter 2: locally compact group

2.1: Topological group

def: G, group, topological space s.t. $G \times G \to G$, $(x,y) \mapsto x \cdot y$; $x \mapsto x^{-1}$, $G \to G$ } both continuous

Example: $(\mathbb{R}^n, +)$, $\mathbb{R}_x = ((0,\infty), \times)$

$GL_n(\mathbb{R})$, SL_n, $\mathbb{T} = \mathbb{R}/\mathbb{Z} \sim S^1$ } both are Lie groups

↳ we may skip most content about Lie group

Notation: Denote by 1, the identity

$\forall x, y \in G$, $A, B \subseteq G$, denote $x \cdot A \overset{def}{=} \{xa : a \in A\}$, similarly $B \cdot y$

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}$$

$$A^{-1} = \{a^{-1} : a \in A\}$$

neighborhood, compact neighborhood: open with compact closure.
locally compact

Prop 2.1: (a) Translations and inversion are homeomorphism

Moreover U is open $\Rightarrow U \cdot A$ is open, $\forall A \subset G$ ✓ skip the proof

(b) $\forall$ neighborhood U of 1, $\exists$ a symmetric neighborhood V of 1 s.t. $V \cdot V \subset U$

proof of (b): $(x,y) \mapsto x \cdot y$ is continuous

$W \times W \in$ pre-image of U, then take $V = W \cap W^{-1}$ (Common Argument)

(c) If H is a subgroup of G, so is $\overline{H}$

proof of (c): $\forall \bar{x}, \bar{y} \in \overline{H}$, consider $\bar{x} \cdot \bar{y}$

$\forall$ neighborhood U of $\bar{x} \cdot \bar{y}$, by continuity of $(x,y) \mapsto x \cdot y$, $\exists$ neighborhood W_1 of $\bar{x}$

and neighborhood W_2 of $\bar{y}$, s.t. $W_1 \cdot W_2 \subset U$

Similarly for $\bar{x}^{-1}$.

(d) Every open subgroup of G is closed. ✓ by considering coset.

(e) A, B compact $\Rightarrow AB$ compact ✓ compactness is preserved by continuity

Suppose H is a subgroup of G, then $q: G \to G/H$, with quotient topology

i.e. $V \subseteq G/H$ is open if and only if $q^{-1}(V)$ is open in G.

Notice that q is an open mapping, $q^{-1}(q(U)) = U \cdot H$ is open if U is open.

Proposition 2.2 (a) If H is closed, then G/H is Hausdorff.

(b) If G is locally cpt. so is G/H ⟵ Easy. by quotient mapping, skip the proof.

(c) If H is normal, then G/H is a topological group.

Proof: (a)

$\forall \bar{x} \neq \bar{y} \in G/H$. consider xHy^{-1}, well-defined (closed) and $1 \notin xHy^{-1}$

$\Rightarrow (xHy^{-1})^c$ is a neighborhood of 1

$\Rightarrow \exists$ symmetric U of 1 s.t. $U \cdot U \cap xHy^{-1} = \emptyset$

$\Rightarrow \underbrace{UxH}_{开集} \cap \underbrace{UyH}_{开集} = \emptyset \Rightarrow$ Hausdorff.

(c): It's easy to see that G/H is a group., only need to show that multiplication & inverse are continuous.

Now we show that $(\bar{x}, \bar{y}) \to \bar{x}\bar{y}$ is continuous.

$\forall$ neighborhood U of $\bar{x}\bar{y} \to q^{-1}(U)$ is a neighborhood of $x \cdot y$

$\Rightarrow \exists$ a neighborhood W_1 of x, neighborhood W_2 of y such that

$W_1 \cdot W_2 \subset q^{-1}(U) \Rightarrow q(W_1 \cdot W_2) \subset U$

$\quad = q(W_1) \cdot q(W_2)$, as $q(x) \cdot q(y) = q(x \cdot y)$

For inversion, $\forall$ neighborhood U of $q(x)$, notice $q(q^{-1}(U)^{-1}) = U^{-1}$ $\boxed{//}$

As $(q^{-1}(U))^{-1} = (U \cdot H)^{-1} = H \cdot U^{-1} = U^{-1} \cdot H$ by normed.

Corollary 2.3: ① If G is T_1 (单点闭集), then G is Hausdorff

proof of ①: $G = G/\{1\}$, by above proposition.

②: If G is not T_1, then $\overline{\{1\}}$ is a closed normal subgroup, and then $G/\overline{\{1\}}$ is a Hausdorff topological group

Proof of ②: First $\overline{\{1\}}$ is the smallest closed subgroup of $G \Rightarrow \overline{\{1\}}$ is normal

then by Prop 2.2 $G/\overline{\{1\}}$ is a Hausdorff topological group. $\boxed{//}$

Prop 2.4: Every locally compact group G has a subgroup G_0, open, closed, σ-compact

Corollary If G is connected, then G is σ-compact

proof of cor: G connected $\Rightarrow G_0 = G$, then G is σ-compact

proof of the prop 2.4:

$\exists$ a symmetric compact neighborhood V of 1, let $G_0 = \bigcup_{n \geq 1} V^n$, subgroup, open, closed.

then $\overline{V^n} \subset G_0 \Rightarrow G_0 = \bigcup_{n\geq 1} \overline{V^n}$, σ-compact! $\boxed{/\!/}$

Def: let f be a function on G. define $L_y f(x) = f(y^{-1}x)$

$$R_y f(x) = f(xy)$$

then $L_{xy} = L_x L_y$, $R_{xy} = R_x R_y$

We say f is left (right) uniformly continuous, if $\|L_y f - f\|_{sup} \to 0$ uniformly as $y \to 1$

$(\|R_y f - f\|_{sup} \to 0)$

Prop 2.6: If $f \in C_c(G)$, then f is left and right uniformly continuous.

similar to the one in mathematics analysis

2.2 Haar measure

Def: A left (right) Haar measure is a non-zero Radon measure on G s.t. $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for all Borel sets $E \subset G$. $\forall x \in G$

finite on cpt sets.

inner (compact) and outer (open) regularity

Example: Lebesgue measure on $(\mathbb{R}^n, +)$

$\mu = \frac{dx}{|x|}$ on $(\mathbb{R}\backslash\{0\}, \times)$, where $\mu(E) = \int \chi_E(x) \cdot \frac{1}{|x|} dx$

$\frac{dxdy}{x^2+y^2}$ on $(\mathbb{C}\backslash\{0\}, \cdot)$

both left and right Haar measure.

Remark: ① If μ is left (right) invariant then $\hat{\mu}(E) \overset{def}{=} \mu(E^{-1})$ is right (left) invariant.

② μ is left haar $\Leftrightarrow \int L_y f(x)\, d\mu(x) = \int f(x)\, d\mu(x)$, $\forall y$, $\forall f \in C_c^+(G)$

积分形式的定义

$$\int L_y f(x)\, d\mu(x) = \int f(y^{-1}x)\, d\mu(x) = \int f(x)\, d(L_y)_* \mu(x)$$

③ $\mu(U) > 0$, if U is an open set of non-empty interior.

proof: we may assume $1 \in U$, then $\mu(U) = 0 \Rightarrow$ every compact set has measure 0

$\Rightarrow$ Every set has measure 0 by inner regularity

$\Downarrow$

contradiction (non-zero measure)

Important! (存在唯一性)

Thm 2.10: Every locally compact group G posses a left Haar measure. Moreover if μ, ν are left Haar measure, then $\mu = c\nu$ for some constant c.

Remark: the same holds for "right".

P45

More examples: $\prod_{i<j} d\alpha_{ij}$ (lebesgue measure) on $\{(\alpha_{ij})_{n\times n} : \alpha_{i,i}=1,\ \alpha_{ij}=0,\ \forall i>j\}$ $\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

on $GL_n(\mathbb{R}) \ni T$, $|\det(T)|^{-n} \prod_{i,j=1}^{n} d\alpha_{ij}$: $T(x_1,\cdots,x_n) = (Tx_1, Tx_2, \cdots, Tx_n)$

left and right invariances.

Now we give an example on which left and right Haar measures are different

$G \overset{\text{def}}{=} \{g : gx = ax+b,\ \forall x\in\mathbb{R},\ \text{for some } a>0,\ b\in\mathbb{R}\} = \{(a,b) \in \mathbb{R}_+ \times \mathbb{R}\}$

then its left Haar measure : $\frac{da\,db}{a^2} = \mu$

right Haar measure $\frac{da\,db}{a} = \nu$,

- In G $(a,b)\cdot(c,d)x$

$= (a,b)(cx+d)$

$= acx + ad + b \rightsquigarrow (a,b)\cdot(c,d) = (ac, ad+b)$

- $1 = (1,0)$, $(a,b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$,

$\boxed{\int L_{(c,d)} f(a,b)\, d\mu}$

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(\underbrace{(c,d)^{-1}(a,b)}_{\substack{\| \\ (\frac{1}{c},-\frac{d}{c})\cdot(a,b) = (\frac{a}{c},\ \frac{b}{c}-\frac{d}{c})}}) \frac{da\,db}{a^2}$

$= \underbrace{\int_{-\infty}^{\infty}\int_0^{\infty} f(\frac{a}{c}, \frac{b}{c} - \cancel{\frac{d}{c}})}_{\text{translation}} \frac{da\,db}{a^2}$

let $a' = \frac{a}{c}$, $b' = \frac{b}{c}$, then

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(a',b') \frac{da'db'}{(a')^2} = \int f\, d\mu \Rightarrow$ left-invariant $\Rightarrow \mu$ is left-invariant.

Now for ν $\int R_{(c,d)} f(a,b)\, d\nu(a,b)$

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(ac, a\cancel{d}+b) \frac{da\,db}{a}$ (translation)

$= \int_{-\infty}^{+\infty}\int_0^{\infty} f(a',b') \cdot \frac{da'db'}{a'} = \int f\, d\nu \Rightarrow \nu$ is right-invariant.

Remark: • Lie group: left-invariant volume form. 不变的测度

• If G_α compact, with probability left Haar measure λ_α, then one can construct (compact 上 left, right 都一样)

a left Haar measure λ on $\prod_\alpha G_\alpha$ by extension of

$$I(f) = \int\cdots\int f(x_{\alpha_1},\cdots x_{\alpha_n})\, d\lambda_{\alpha_1}\cdots d\lambda_{\alpha_n}$$

$f \in C(G)$, $f=1$ for all λ_α, but $\alpha_1,\cdots,\alpha_n$

In particular $\mathbb{Z}_2^{\mathbb{N}}$ equivalently $\mathbb{Z}_2^{\mathbb{N}} \to [0,1]$

$$(a_1, a_2, \cdots) \longmapsto \sum_{j=1}^{\infty} a_j \frac{1}{2^j}.$$