Important example $\boxed{\mathbb{Q}_p}$: the field of p-adic numbers

$\forall r \in \mathbb{Q}\setminus\{0\},\ r = p^m \cdot \frac{a}{b},\ (a,b)=1,\ p \nmid ab$

$\downarrow$

then we define p-adic norm of r. $|r|_p = p^{-m}$. In addition, define $|0|_p = 0$

Notice $|r_1 + r_2|_p \le \max(|r_1|_p, |r_2|_p)$

$|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$

Now $d(r_1, r_2) \overset{\text{def}}{=} |r_1 - r_2|_p$ defines a metric on $\mathbb{Q}$

$\downarrow$

denote by $\mathbb{Q}_p$ its completion, called the field of p-adic members.

Prop 2.8 If $m \in \mathbb{Z}$, $c_j \in \{0, 1, \cdots, p-1\}$ for $j \ge m$, then every sequence $\sum_{j\ge m} c_j p^j$ (级数的展开形式) is convergent in $\mathbb{Q}_p$. ①

moreover, every p-adic number is the sum of such a series

Proof: $|\sum_{j=n}^{N} c_j p^j|_p \le p^{-m} \to 0$, as $n \to \infty$ ($\forall N$), thus the sequence is Cauchy

$\Downarrow$

① holds, convergent in $\mathbb{Q}_p$.

On the other hand, it sufficies to show

$\{\sum_{j=m}^{\infty} c_j p^j,\ m \in \mathbb{Z},\ c_j = \{0, 1, \cdots, p-1\}\} \subseteq \mathbb{Q}_p$ is a ~~field~~ field containing, that is complete under $|\cdot|_p$

then by complete 的最小性

We first show that it is $\boxed{\text{complete}}$ under $|\cdot|_p$

$\forall$ Cauchy sequence $\sum_{j\ge m_n} c_{j,n} p^j$, then $\forall M > 0$, $\exists N$ s.t.

$|\sum_{j\ge m_{n_1}} c_{j,n_1} p^j - \sum_{j \ge m_{n_2}} c_{j,n_2} p^j|_p \le p^{-M}$, $\forall n_1, n_2 > N$

$\Downarrow$

$c_{j,n_1} = c_{j,n_2}$ for all $j \ge N$

$\Rightarrow c_j, j \ge m$, s.t. $\forall j$, $\exists N_1$ s.t. $c_j = c_{j,n}$, $n > N$

$\Rightarrow \sum c_j p^j$ is the limit. $\Rightarrow$ complete.

It remains to show that $\mathbb{Q}_p$ is a field $\boxed{\text{containing } \mathbb{Q}}$.

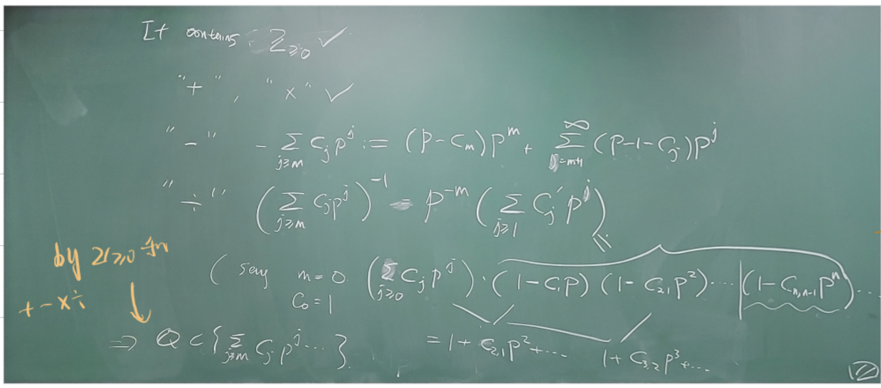
$\downarrow$

It contains $\mathbb{Z}_{\ge 0}$ ✓

"+" ✓ "×" ✓

"−" : $-\sum_{j\ge m} c_j p^j \overset{\text{def}}{=} (p - c_m)p^m + \sum_{j=m+1}^{\infty} (p-1-c_j) p^j$

▲ "÷" : $(\sum_{j\ge m} c_j p^j)^{-1} = p^{-m} (\sum_{j\ge 1} c_j' p^j)$



by $\mathbb{Z}_{\ge 0}$ 和 $+ - \times \div$

the only completion besides $\mathbb{R}$

Moreover ①: $|x|_p = p^{-k}$, discreate, $\Rightarrow$ each ball $B(\overset{\text{radius}}{r}, \overset{\text{center}}{x})$ is both open and closed.

quite important later.

② Since $|x-y|_p \le \max\{|x|_p, |y|_p\}$, we have

$|x-y|_p < r$, $|y-z|_p < r$, then $|x-z| < r$.

$\Rightarrow$ Every element in a ball is the center of this ball.

(First $B(r,y) \subset B(r,x)$, $\forall x \in B(r,y)$, then $B(r,x) \subset B(r,y)$ 对称 $\Rightarrow B(r,x) = B(r,y)$)

Corollary: Every 2 balls are either disjoint or one contains another.

③ Every $B(p^m, x)$ contains exactly p^{m-n} balls of radius p^n $(n<m)$

$\| \; x + \sum_{j\ge -m} c_j p^j$ $\qquad$ something $+ \sum_{j\ge -n} c_j p^j$

$\| \; \underbrace{x + \sum_{j=-m}^{-n+1} c_j p^j}_{p^{m-n} \text{ options}} + \sum_{j\ge -n} c_j p^j$

③ $\Rightarrow$ Every bounded sequence has a convergent subsequence → sequentially compact.

$\Rightarrow$ $\mathbb{Q}_p$ is locally compact (完备度量空间, compactness 多 criteria 等价)

④ $(\mathbb{Q}_p, +)$, $(\mathbb{Q}_p \setminus \{0\}, \times)$, locally compact topological group.

For its Haar measure on $(\mathbb{Q}_p, +)$, say $\overset{\text{Haar measure}}{\lambda}(\underset{\text{the integer ring}}{B(0,1)}) = 1$, then every ball of radius p^m has

measure measure p^m, then by outer regularity

$$\lambda(E) = \inf\{\sum p^{m_j} : E \subset \cup_1^\infty B(p^{m_j}, x_j)\}$$

Section 2.4: The modular functions

let λ be left (WLOG) Haar measure on G: $\begin{cases} \lambda(xE) = \lambda(E) \\ \int L_y f \, d\lambda = \int f \, d\lambda \end{cases}$ (left translation by y)

$\| \; \boxed{\int f \, d\underbrace{((L_y)_* \lambda)}_{\text{measure}}} = \int f(y^{-1}x) \, d\lambda(x)$

Question: What is $\int f(xy) \, d\lambda(x)$?

$\| \; \int R_y f \, d\lambda = \int f \, d((R_y)_* \lambda)$

For each y, $\int f(zxy) \, d\lambda(x) = \int f(xy) \, d\lambda(x)$

$(R_y)_* \lambda$ is left-invariant

now by uniqueness, $(R_y)_* \lambda = \underbrace{\Delta(y^{-1})}_{\text{constant}} \lambda$

In fact $\Delta(y^{-1})$ is independent in the choice of λ

$$\boxed{\int f(xy)\,d\lambda(x) = \Delta(y^{-1})\int f(x)\,dx}$$

To see this, fix λ_0, then $\forall \lambda$, $\lambda = c_\lambda \cdot \lambda_0$. Say $\Delta(y^{-1})$ is given by λ_0, then $(R_y)_*\lambda = c_\lambda (R_y)_*\lambda_0$

$$= c_{\lambda_0}\cdot \Delta(y^{-1})\lambda_0$$

$$= \Delta(y^{-1})\cdot\lambda$$

Prop 2.4: Δ (the modular function of G) is a continuous homomorphism from G to $\mathbb{R}^\times = (\mathbb{R}_+, \times)$

proof: $\Delta(1) = 1$, $\Delta(yz)\int f(x)\,d\lambda(x)$

$$= \int \underbrace{f(xz^{-1}y^{-1})}_{=F(xz^{-1})}\,d\lambda(x), \text{ denote } F(x) = f(xy^{-1})$$

$$= \Delta(z)\int F(x)\,d\lambda(x)$$

$$= \Delta(z)\cdot\Delta(y)\int f(x)\,d\lambda(x) \quad \Rightarrow \Delta(yz) = \Delta(y)\cdot\Delta(z), \text{ so group homomorphism} \to \Delta(x^{-1}) = \Delta(x)^{-1}$$

For continuity, recall $\forall f \in C_c(G)$, then $|R_y f - R_{y_0} f| \to 0$, uniformly as $y \to y_0$

$$\Rightarrow \underbrace{\int R_y f(x)\,d\mu(x)}_{=\Delta(y^{-1})\int f} \to \underbrace{\int R_{y_0} f(x)\,d\mu(x)}_{=\Delta(y_0^{-1})\int f} \qquad \blacksquare$$

def: G is called uni-modular if $\Delta(x) \equiv 1$, in which case, left Haar is also right Haar.

e.g. Abelian group is unimodular. ✓

Prop 2.27: If K is a compact, then $\Delta|_K \equiv 1$

proof: $\Delta(K)$ is a compact subgroup of $\mathbb{R}^\times \Rightarrow \Delta(K) = \{1\}$. $\blacksquare$

Corollary: Every compact group is unimodular.

Prop 2.29: $[G, G] \overset{\text{def}}{=} \{xyx^{-1}y^{-1} : x, y \in G\}$ denoted by

If $G/[G,G]$ is compact, then G is uni-modular.

Proof: $\Delta(xyx^{-1}y^{-1}) = 1$, then Δ on G induces a Δ on $G/[G:G]$, which must be 1. $\blacksquare$

Recall that λ left Haar $\Leftrightarrow$ ρ right Haar

$$\rho(E) = \lambda(E^{-1}), \quad \int f(x)\,d\rho(x) = \int f(x^{-1})\,d\lambda(x)$$

Prop 2.31: $\int f(x)\,d\rho(x) = \int f(x) \boxed{\Delta(x^{-1})\,d\lambda(x)}$

i.e. $d\lambda(-x) = \Delta(x^{-1})\,d\lambda(x)$

Proof: We first show that $\Delta(x^{-1})\,d\lambda(x)$ is right-invariant

$$\int R_y f(x)\,\underbrace{\Delta(x^{-1})}_{=\Delta(x)^{-1} = \Delta(xyy^{-1})^{-1}}\,d\lambda(x) = \int R_y\big(f(x)\cdot\Delta(yx^{-1})\big)\,d\lambda(x)$$

$$= \Delta(y^{-1})\int f(x)\,\Delta(yx^{-1})\,d\lambda(x) = \int f(x)\,\Delta(x^{-1})\,d\lambda(x)$$

then $\Delta(x^{-1})\,d\lambda(x) = c\cdot d\rho(x)$

$\forall\, U \subseteq G$, symmetric compact neighborhood of 1 $\Rightarrow \lambda(U) = \rho(U) > 0$

$\Rightarrow (c-1)\lambda(U) = c\rho(U) - \lambda(U) = \int_U \underbrace{(c\Delta(x^{-1})-1)}_{\le \varepsilon}\,d\lambda(x) \le \varepsilon\lambda(U)$

$\le \varepsilon$, when U is "small enough" as Δ is continuous.

$\Rightarrow c = 1$. $\blacksquare$

Remark: If G is not unimodular, Δ is not bounded

so $f(x) \mapsto f(x^{-1})$ is not isometry in $L^p(\lambda)$ $(1 \le p < \infty)$

However, now we have 2 ways to construct isometries between $L^p(\lambda)$ and $L^p(\rho)$

$L^p(\lambda)$: (1) $f(x) \mapsto f(x^{-1})$ $L^p(\rho)$ 可逆的isometry

$\left(\int f(x^{-1})\,d\rho(x) = \int f(x)\,d\lambda(x)\right)$

(2) $f(x) \mapsto \Delta(x)^{\frac{1}{p}} f(x)$

$\int \Delta(x)\cdot|f(x)|^p\,d\rho(x) = \int \Delta(x)|f(x)|^p \overset{\text{prop 2.3)}}{\Delta(x^{-1})}\,d\lambda(x) = \int |f(x)|^p\,d\lambda(x)$

by combining (1), (2), we obtain an isometry on $L^p(\lambda)$

$f(x) \to \Delta(x^{-1})^{\frac{1}{p}} f(x^{-1})$

In particular, $\int f(x)\,d\lambda(x) = \int f(x^{-1})\Delta(x^{-1})\,d\lambda(x)$, p取1