We finish chapter 2 (Locally compact groups), but will not cover everything
↑ since we mainly deal with abelian group with good properties.

2.5: Convolutions (G locally compact, λ left Haar measure), dx for convenience

Recall in Real analysis: $\forall f, g \in L^1(\mathbb{R}^d)$, • $f*g = \int f(y)\, g(x-y)\, dy \in L^1$
$= g*f$. • If take $g = \phi_\varepsilon$ (approximate identity), then $f*\phi_\varepsilon \to f$, as $\varepsilon \to 0$

• $\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$ $\qquad f \in L^p,\ g \in L^{p'}$, then $f*g$ is continuous.

$\Rightarrow$ $\|f*g\|_{L^\infty} \le \|f\|_{L^p} \|g\|_{L^{p'}}$

Now in locally compact group

Def: $\forall f, g \in L^1(G)$, def
$$f*g\,(x) = \int f(y)\, g(y^{-1}x)\, dy \in L^1,$$
$$\forall \phi \in C_c(G),\ \int \phi \cdot f*g = \iint \phi(x) f(y)\, g(y^{-1}x)\, dx\, dy$$
left invariant
$$= \iint \phi(yx) f(y)\, g(x)\, dx\, dy$$

However, in general, G $f*g \ne g*f$

LHS: $\int f(y)\, g(y^{-1}x)\, dy$ $\qquad$ RHS: $\int g(y)\, f(y^{-1}x)\, dy$

$= \int f(xy)\, g(y^{-1})\, dy$ $\qquad$ "=" if G is abelian

$= \int f(xy^{-1})\, g(y)\, \Delta(y^{-1})\, dy$ $\qquad$ "≠" even if G is unimodular

Observation: $L_z(f*g)(x) = \int f(y)\, g(y^{-1}zx)\, dy = \int f(zy)\, g(y^{-1}x)\, dy$
$$= (L_z f)*g$$

also $R_z(f*g)(x) = \int f(y)\, g(y^{-1}xz)\, dy = f*(R_z g)$

some details may be omitted.

Prop 2.40, $1 \le p \le \infty$, $f \in L^1$, $g \in L^p$, then

$\boxed{(a)}$ $f*g \in L^p$, $\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$

(b) If G is unimodular, (a) holds for $g*f$

(c) If G is not-unimodular, then (a) holds for $g*f$, if f has compact support.

Proof: (a) $\|f*g\|_{L^p} = \left(\int \left|\int f(y)\, g(y^{-1}x)\, dy\right|^p dx\right)^{\frac{1}{p}}$ $\quad \| \|\cdot\|_{L^1} \|_{L^p}$

Minkowski
$$\le \int \left(\int |f(y)|^p\, |g(y^{-1}x)|^p\, dx\right)^{\frac{1}{p}}$$
$$= \|f\|_{L^1} \cdot \|g\|_{L^p}$$

(b) Similar by Minkowski, $g*f = \int g(y)\, f(y^{-1}x)\, dy = \int g(xy^{-1})\, f(y)\, \underset{=1}{\Delta(y^{-1})}\, dy$, then Minkowski

(c): $|\Delta(y)| \approx 1$, $y \in \operatorname{supp} f$.

Prop 2.4.1: Suppose G is unimodular, $f\in L^p$, $g\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$, then

$f*g\in C_0(G)$, and $\|f*g\|_{sup}\le\|f\|_{L^p}\|g\|_{L^q}$

vanish at ∞

$\forall\varepsilon, \exists$ compact $K\subset G$ s.t. $|f|<\varepsilon$ outside K

Proof: ① $\|f*g\|_\infty\le\|f\|_{L^p}\cdot\|g\|_{L^q}$ by Hölder

② Approximate f, g by C_c functions. $\square$

Recall in $\mathbb{R}^n$, $\|f(x+\cdot)-f(\cdot)\|_{L^p}\to0$, as $x\to0$, if $f\in L^p$ (L^p-continuity)

In G.

Prop 2.4.2: $1\le p<\infty$, $f\in L^p(G)$, then $\|L_yf-f\|_{L^p}\to0$ also by uniformly continuity.

$\|R_yf-f\|_{L^p}\to0$, as $y\to1$

approximate f, g by C_c functions

Prop 2.4.3: $f\in L^1$, $g\in L^\infty$, then $f*g$ is left uniformly continuous.

$g*f$ is right uniformly continuous.

Proof of 2.4.2: Omit

Proof of 2.4.3: recall $L_y(f*g)=(L_yf)*g$, now

$R_y(g*f)=g*(R_yf)$

$$L_y(f*g)(x)-f*g(x)=(L_yf)*g-f*g(x)$$

$$=(L_yf-f)*g,\quad\text{take }\tilde{f}\in C_c(G),\text{ s.t. }\|f-\tilde{f}\|_{L^1}<\varepsilon$$

$$=(L_y\tilde{f}-\tilde{f})*g(x)+O(\varepsilon)$$

$$\le\|L_y\tilde{f}-\tilde{f}\|_{L^1}+O(\varepsilon)=O(\varepsilon).\quad\square$$

R_y of the similar principle.

When G is discrete, $\delta(x)=\begin{cases}1, & x=0\\ 0, & \text{elsewhere}\end{cases}\in C_c(G)$, and $f*\delta(x)=f(x)$

$\| \int f(y)\,\delta(y^{-1}x)\,dy$

Haar measure on discrete set $\Rightarrow$ counting measure (up to a constant)

For G, general group, a function δ s.t. $f*\delta=f$ might NOT exist!

the following prop (approximating identity)

Prop 2.4.4 (Approximate identity)

Let $\mathcal{U}$ be a neighborhood base at 1. For each $U\in\mathcal{U}$, let ψ_U be a L^1-function s.t.

(i) supp ψ_U is compact

(ii) $\psi_U\ge0$, and $\int\psi_U=1$

then $\|\psi_U*f-f\|_{L^p}\to0$, as $U\to\{1\}$, if $f\in L^p$, $1\le p<\infty$, or f right uniformly continuous, $p=\infty$

If, in addition, $\psi_U(x^{-1}) = \psi_U(x)$, then the above holds for $\|\underbrace{f * \psi_U}_{\text{换序了!}} - f\|_{L^p}$ - - -

Proof: $\psi_U * f(x) - f(x) = \int \psi_U(y) \underbrace{f(y^{-1}x)}_{= L_{y^{-1}} f(x)} dy - \underbrace{f(x)}_{= \int \psi_U(y) f(x) dy}$

$$= \int \psi_U(y) \left(L_{y^{-1}} f(x) - f(x) \right) dy, \text{ then}$$

$$\|\psi_U * f - f\|_{L^p} = \left(\int \left(\int \psi_U(y) (L_{y^{-1}} f(x) - f(x)) dy \right)^p dx \right)^{\frac{1}{p}}$$

$$\overset{\text{minkowski}}{\leq} \int \underbrace{\|L_{y^{-1}} f(x) - f(x)\|_{L^p}}_{\to 0, \text{ as } y \to 1} \psi_U(y) dy$$

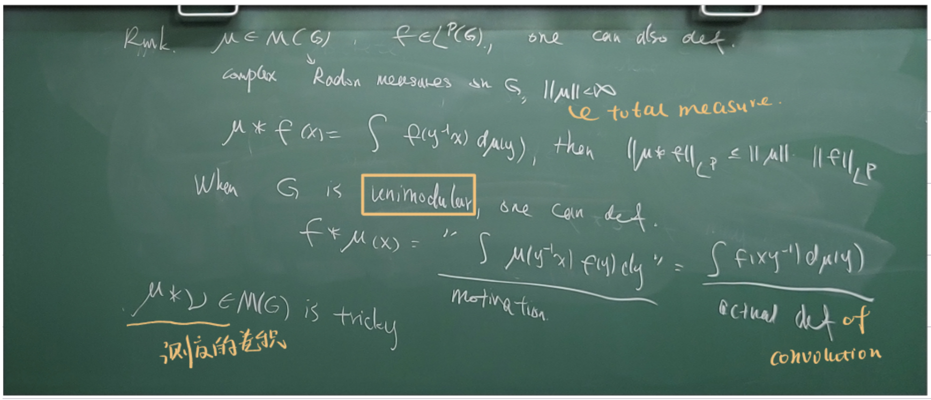
$$\leq \varepsilon \cdot \int \psi_U(y) dy = \varepsilon. \quad \text{as } U \to \{1\}.$$

If $\psi_U(x^{-1}) = \psi_U(x)$, $\underbrace{f * \psi_U(x)}_{\text{换序}} - f(x)$

$$= \int f(y) \underbrace{\psi_U(y^{-1}x)}_{= \psi_U(x^{-1}y)} dy - f(x) = \int \psi_U(y) \left(R_y f(x) - f(x) \right) dy$$

then apply Minkowski ▨

Remark:



Section 2.6: Homogeneous Spaces

H: closed subspaces, $G \circlearrowright G/H$ homogeneous space, $\underbrace{G\text{-space}}_{\text{A space } S \text{ equipped with an action of } G}$ (group action)

Model (G-space): $G \circlearrowright \underbrace{S}_{\text{locally compact Hausdorff}}$

fix $s_0 \in S$, and let $\underbrace{H \overset{\text{def}}{=} \{x \in G : x s_0 = s_0\}}_{\text{closed}}$

then consider G/H, and when the action is $\underline{\text{transitive}}$ $\forall s_0, s_1 \in S$, $\exists\, x \in G$, $x s_0 = s_1$

then $\Phi : G/H \to S$ is a continuous bijection.

It may $\boxed{\text{not}}$ be a homeomorphism, $\underbrace{\mathbb{R}}_{\text{discrete topology}} \circlearrowright \underbrace{\mathbb{R}}_{\text{regular topology}}$

Prop 2.46: Φ is a homeomorphism when $\underbrace{G \text{ is } \sigma\text{-compact}}_{S \text{与} G/H \text{同胚}}$.

skip the proof.

Goal: 有闭子群上的积分, G上的积分 $\Rightarrow$ 表达出 G/H上的积分

Def: $\forall f \in C_c(G)$, define $\overset{\text{projection}}{Pf}(xH) \overset{\text{def}}{=} \boxed{\int_H f(x\xi)\, d\xi \in C_c(G/H)}$

main theorem

Thm 2.5.1: $\exists$ G-invariant measure μ on $G/H \iff \Delta_G|_H = \Delta_H$,

Moreover, in this case μ is unique up to a constant factor, and if this factor is suitably chosen, we have

$$(*) \quad \int_G f(x)\,dx = \int_{G/H} Pf\, d\mu = \int_{G/H}\int_H f(x\xi)\,d\xi\, d\mu(xH), \quad \forall f \in C_c(G)$$

Remark: ①: It holds when H is compact

②: It holds for $f \in L^1$, or $f \geq 0$, supp f is σ-finite
by approximation of cpt-supp func (e.g. in measure theory)

③: It can be reduced to "Fubini" when G is second countable
see Notes and preferences.

proof: the key of the proof is to show that all $C_c(G/H)$ function can be written as Pf, for some $f \in \overset{\text{compact!}}{C_c}(G)$

Lemma 2.4.8 (提升引理): If $E \subset G/H$ is compact, $\exists K \subset G$, compact. s.t. $\overset{\text{canonical quotient map: } G \to G/H}{q}(K) = E$

proof: Take a compact neighborhood V of 1 in G, then $E \subset \bigcup_{x \in q^{-1}(E)} q(xV)$, open cover

$\Rightarrow E \subset \bigcup_{j=1}^{n} q(x_j V)$ by locally compact, finite open cover

take $K = q^{-1}(E) \cap (\bigcup_{j=1}^{n} x_j \bar{V})$. ▯

Lemma 2.49 (提升函数(indicator function)): If $F \subset G/H$ compact, $\exists f \in C_c(G)$, $f \geq 0$, s.t. $Pf|_F = 1$

proof: take $\phi \in C_c(G/H)$, $\phi = 1$ on F, and $g \in C_c(K)$, where K is the compact set from Lemma 2.48 $q(K) = F$

let $f = \dfrac{\phi \circ q}{Pg \circ q} \cdot g$, then $\forall xH \in F$,

$$Pf(xH) = \int_H f(x\xi)\,d\xi = \frac{\phi(xH)}{Pq(xH)} \underbrace{\int_H g(x\xi)\,d\xi}_{= Pg(xH)} = \phi(xH)$$

▯

Proposition 2.50: If $\phi \in C_c(G/H)$, then $\exists f \in C_c(G)$ s.t. $Pf = \phi$, $q(\text{supp} f) = \text{supp}(\phi)$, and $f \geq 0$, if $\phi \geq 0$

proof: $f = (\phi \circ q) \cdot \underbrace{g}_{\text{from Lemma 2.49}}$

then $Pf = \phi \cdot \underbrace{Pg}_{=1 \text{ on supp } \phi}$, other properties of f are obvious. ▯

Now. proof of thm 2.5.1

Proof: Suppose $\exists$ a G-invariant measure μ on G. define $f \mapsto \int Pf(xH)\, d\mu(xH)$ $\forall f \in C_c(G)$

left-invariant: $L_y f \mapsto \int Pf(yxH)\,d\mu(xH) = \int Pf(xH)\,d\mu(xH)$

by uniqueness of Haar measure $\int f\,dx = c\cdot \int Pf\,d\mu$ $\to$ uniquely determined by lemma 2.5 ($\uparrow$ once we choose c)

one can take $c=1$, $\int f\,dx = \int Pf\,d\mu$

$$= \int_{G/H} \int_H f(x\xi)\,d\xi\,d\mu(xH)$$

Then $\Delta_G(\eta)\int_G f(x)\,dx \overset{\text{by def of }\Delta}{=} \int_G f(x\eta^{-1})\,dx$, $\forall \eta\in H$

$$= \int_{G/H} \underbrace{\int_H f(x\xi\eta^{-1})\,d\xi}\,d\mu$$

$$= \Delta_H(\eta)\underbrace{\int_{G/H}\int_H f(x\xi)\,d\mu}_{=\int_G f} \implies \Delta_G(\eta) = \Delta_H(\eta),\ \forall \eta\in H.$$

Conversely $\Leftarrow$: assume $\Delta_G|_H = \Delta_H$, We need to define a positive linear functional on G/H

Def: We have proved every $C_c(G/H)$ function can be written as Pf

we would like to define $Pf \longmapsto \int_G f\,dx$ on $C_c(G/H)$, $f\in C_c(G)$

G invariant ✓ positive ✓

It remains to show that it is well-defined, i.e. $\underset{\int_H f(x\xi)\,d\xi}{\underset{\|}{Pf}}=0 \implies \int_G f = 0$, $f\in C_c(G)$

By Lemma 2.49, $\exists\, \phi\in C_c(G)$ s.t. $P\phi = 1$ on $\underbrace{q(\operatorname{supp} f)}_{\text{cpt}}$ $\Rightarrow \int_H \phi(x\xi)\,d\xi = 1$ on supp f

$$0 = \int_G \phi(x)\int_H f(x\xi)\,d\xi\,dx \overset{\text{Fubini}}{=} \int_H\int_G \phi(x) f(x\xi)\,dx\,d\xi$$

$$= \int_H \Delta_G(\xi^{-1})\int_G \phi(x\xi^{-1}) f(x)\,dx\,d\xi$$

$\|$ by condition

$$= \int_H \Delta_H(\xi^{-1})\int_G \phi(x\xi^{-1}) f(x)\,dx\,d\xi$$

$$= \int_H \int_G \underline{\phi(x\xi)}\, f(x)\,dx\,d\xi$$

$$= \int_G f(x)\int_H \phi(x\xi)\,d\xi\,dx = \int_G f(x)\,dx.$$