partial collection of chapter 1 and chapter 3 $\Rightarrow$ "五一"后直接 chapter 4 (locally compact abelian group上的 Fourier 分析)

如果需要更复杂的结论再补!

Chapter 1. Banach Algebra and spectrum theorem

rather deep theory

Def: Banach algebra $\mathcal{A}$ over $\mathbb{C}$ is an algebra with a norm $\|\cdot\|$ that makes it a Banach space with $\|x\cdot y\| \le \|x\|\cdot\|y\|$.

- Unital Banach algebra: $\exists\, e$
- Involution on $\mathcal{A}$: an automorphism $\mathcal{A}\to\mathcal{A}$ such that

  e.g. 矩阵转置, transpose

$$(\lambda x)^* = \bar{\lambda} x^*, \quad x^{**} = x$$

$$(x+y)^* = x^* + y^*, \quad (x\cdot y)^* = y^*\cdot x^*$$

(Banach) $*$-algebra = Banach algebra + involution

C^*-algebra = $*$-algebra + $\|x\cdot x^*\| = \|x\|^2$

$\Rightarrow \|x\| = \|x^*\|$, for $\|x\|^2 = \|x\cdot x^*\| \le \|x\|\cdot\|x^*\|$

$\|x^*\|^2 = \|x^*\cdot (x^*)^*\| = \|x^*\cdot x\| \le \|x\|\cdot\|x^*\|$

Homomorphism: same in algebra

$*$-Homomorphism: Homomorphism + $\phi(x^*) = \phi(x)^*$

Example: X compact Hausdorff

- $C(X)$ is unital C^*-algebra, 有identity, $f\cdot g$, $e = 1_X$, $\|f\| = \sup|f|$

$$f^* = \bar{f}, \quad f * f^* = |f|^2$$

- If X is not compact, then $C(X) \overset{\text{def}}{=} \{f \text{ continuous}, \|f\|_{\sup} < \infty\}$, $C(X)$ is still unital but $C_0(X)$ is not unital.

  vanishing at ∞

Example 2: H Hilbert space

$\mathcal{L}(H) = \{\text{bounded linear operators on } H\}$, unital C^*-algebra

Example 3: $L^1(G)$: $*$-algebra, not C^*. $\|f*g\|_{L^1} \overset{\le}{\ne} \|f\|_{L^1}\cdot\|g\|_{L^1}$

unital if and only if G is discrete.

$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ to ensure $(f*g)^* = g^* * f^*$

From non-unital to unital. suppose $\mathcal{A}$ is non-unital

Construct $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$, with $(x,a)\cdot(y,b) = (xy + ay + bx, ab)$, and

$$\|(x,a)\| = \|x\| + |a|$$

In this case, $e = (0,1)$, and $\mathcal{A} \simeq \mathcal{A} \times \{0\}$ is a closed (maximal) ideal of $\tilde{\mathcal{A}}$

If $\mathcal{A}$ is $*$-algebra, so is $\tilde{\mathcal{A}}$ with $(x,a)^* = (x^*, \bar{a})$.

Example 1:

补充的 identity → unital ; total variation

$\mathcal{A} = L^1(\mathbb{R})$, $\tilde{\mathcal{A}} = \text{Span}\{\mathcal{A}, \delta\} \subseteq M(\mathbb{R}) = \{\mu\}$, $\|\mu\| = |\mu|(\mathbb{R})$

finite Borel measure.

$(f, a) = f + a\delta$, $\|(f,a)\| = \|f\|_1 + |a|$

Example 2:

$C_0(X)$, X is not compact. consider $\tilde{\mathcal{A}} = \underbrace{\text{span}\{\mathcal{A}, 1_X\}}_{\text{Indeed a unital algebra } = C(\hat{X})} \subset C(X)$

one-pt compactification!

现在，范数的选取上会有一点问题！

参考 Example 1 above

$\forall f \in \tilde{\mathcal{A}}$, $\|f\| = \|f - f(\infty)\|_{\sup} + |f(\infty)| \neq \|f\|_{C(X)}$

不等于 Embed 空间的 norm! $\Rightarrow$ Not a severe problem

Example 1 的选取的 norm 太粗糙

$\Downarrow$

the following prop

So $\mathcal{A}$ is $C^* \not\Rightarrow$ so is $\tilde{\mathcal{A}}$ ($\|x \cdot x^*\| = \|x\|^2$)

Proposition 1.27: If $\mathcal{A}$ is a non-unital C^*-algebra, $\exists!$ norm on $\tilde{\mathcal{A}}$ that makes $\tilde{\mathcal{A}}$ a C^*-algebra and this norm agrees with the original norm on $\mathcal{A}$.

a little complicated.

proof: define $\|(x,a)\| \overset{\text{def}}{=} \sup\{ \underset{\|xy+ay\|}{\underset{\|}{\|(x,a)\cdot(y,0)\|}} : y \in \mathcal{A}, \|y\| \le 1\}$. $\cdots$ $\blacksquare$

Now $\mathcal{A}$ is unital, then we can discuss x^{-1}

Simple facts: $\|x\| < 1 \Rightarrow (e - x)^{-1} = \sum_{n=0}^{\infty} x^n$

Corollary 1. If $|\lambda| > \|x\|$, then $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n$ (rescaling)

2. $\|y\| \cdot \|x^{-1}\| < 1 \Rightarrow (x - y)^{-1} = x^{-1}\sum_{n=0}^{\infty}(yx^{-1})^n$

$\underset{x^{-1}(e - yx^{-1})^{-1}}{\|}$

取逆 being continuous

$y = \delta x \Rightarrow$ continuous

3. $\|y\|\cdot\|x^{-1}\| \le \frac{1}{2} \Rightarrow \|(x-y)^{-1} - x^{-1}\|$

$\|$

by (2). $\|x^{-1}\cdot\sum_{n=1}^{\infty}(yx^{-1})^n\| = \|\underbrace{yx^{-2}\sum_{n=0}^{\infty}(yx^{-1})^n}_{\text{bounded}}\|$

$\to 0$, as $\|y\| \to 0$

(1), (2), (3) $\Rightarrow$ $\{x$ is invertible$\}$ is open, and $x \mapsto x^{-1}$ is continuous.
(3)

Def: $\forall x \in \mathcal{A}$ unital, the spectrum of x is defined by $\sigma(x) = \{\lambda : \lambda e - x$ is not invertible$\}$

closed (∵之前 invertible 是 open)
and $\lambda \to \lambda e - x \in \mathcal{A}$ is continuous.

Also $\sigma(x) \subset B_{\|x\|}$.

Def: For $\lambda \notin \sigma(x)$, $R(\lambda) \overset{\text{def}}{=} (\lambda e - x)^{-1}$ is called the resolution element of x

Lemma 1.5: $R(\lambda)$ is $\boxed{\text{analytic}}$ in $\mathbb{C} \setminus \sigma(x)$

$R'(\lambda)$ exists, or $\forall \phi \in \mathcal{A}^*$, $\phi \circ R(\lambda)$ is analytic.

bounded linear functional

Proof: $\forall \lambda, \mu \notin \sigma(x)$

Calculate $\boxed{\lim_{\mu\to\lambda} \frac{R(\mu)-R(\lambda)}{\mu-\lambda}}$, $(\mu-\lambda)e = (\mu e - x) - (\lambda e - x)$

$$= \underbrace{(\lambda e - x)R(\lambda)}_{e}(\mu e - x) - (\lambda e - x)R(\mu)(\mu e - x)$$

$$= (\lambda e - x)(R(\lambda) - R(\mu))(\mu e - x)$$

$$\Rightarrow \frac{R(\lambda)-R(\mu)}{\mu-\lambda} = -R(\lambda)R(\mu) \Rightarrow R'(\lambda) = -R(\lambda)^2.$$

▨

Proposition 1.6: $\sigma(x)$ is $\boxed{\text{non-empty}}$, $\forall x$

Proof: $R(\lambda) \to 0$, as $|\lambda| \to \infty$. So if $\sigma(x) = \phi$, then $\phi \circ (R(\lambda))$ is bounded analytic

$\Rightarrow \phi \circ (R(\lambda)) =$ constant $= 0$.

$\Downarrow$

Contradiction as $\forall y$, $\exists \phi$ s.t. $\phi(y) \ne 0$, by e.g. Hahn-Banach ▨

Theorem 1.7: If every non-zero element in $\mathcal{A}$ is invertible, then $\mathcal{A} \cong \mathbb{C}$

proof: by prop 1.6, $\forall x \in \mathcal{A}$, $\exists \lambda \in \mathbb{C}$, s.t. $\boxed{\lambda e - x \text{ is not invertible}}$

by our assumption, $\forall$ non-zero element is invertible $\Rightarrow \lambda e = x$

$\Rightarrow \mathcal{A} = \mathbb{C}\cdot e$, $\mathcal{A} \cong \mathbb{C}$ ▨

Def: the spectral radius of x is $\rho(x) \overset{\text{def}}{=} \sup\{|\lambda| : \lambda \in \sigma(x)\}$

Thm 1.8: $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$

Application ① $f \in C(X)$ (compact), $\|f\|_{\sup} = \lim_{n\to\infty} \|f\|_{L^n}$

对于不同的空间

② $f \in L^1(\mathbb{R}^d)$, $\lim_{k\to\infty} \|\underbrace{f * \cdots * f}_{k}\|_{L^1}^{\frac{1}{k}} = \|\hat{f}\|_{\sup}$

Will be used in chapter 4

The above are mainly all about the Banach algebra in this class.

Chapter 3 is about representation theory (Basic $\underbrace{\text{representation theory}}_{\text{mainly unitary in this book}}$)

G: locally compact group

$\mathcal{H}_\pi$: Hilbert space

Unitary representation, continuous homomorphism $\pi: G$ (群) $\to$ $\underbrace{U(\mathcal{H}_\pi)}_{\text{unitary operators}}$ (也是群)

$$\pi(xy) = \pi(x)\cdot\pi(y)$$

$$\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*, \quad \dim(\pi) \overset{\text{def}}{=} \dim(\mathcal{H}_\pi)$$

Example: $\mathcal{H}_\pi = L^2(G)$, $\pi_L(x)f(y) = f(x^{-1}y)$, then (Left)

$$(\pi_L(x)f, \pi_L(x)g) = \int f(x^{-1}y)\, g(x^{-1}y)\, dy$$
$$= (f, g) \quad \text{Left-invariant}$$

Left-regular representation.

$\pi_R(x)f(y) \overset{\text{def}}{=} f(x^{-1}y)\cdot\Delta(x)^{\frac{1}{2}}$, right regular representation

Def: $\pi_1: G \to \mathcal{H}_{\pi_1}$, $\pi_2: G \to \mathcal{H}_{\pi_2}$, we say π_1, π_2 are equivalent, if $\exists$ unitary $U: \mathcal{H}_{\pi_2} \to \mathcal{H}_{\pi_1}$

$$\pi_1 \circ U = U \circ \pi_2$$

$$\begin{tikzcd} \mathcal{H}_{\pi_2} \arrow[r, "\pi_2(x)"] \arrow[d, "U"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_{\pi_2} \arrow[d, "U"] \\ \mathcal{H}_{\pi_1} \arrow[r, "\pi_1(x)"] & \mathcal{H}_{\pi_1} \end{tikzcd} \quad \forall x$$

commutative!

More generally, one can consider

$$\mathcal{C}(\pi_1, \pi_2) \overset{\text{def}}{=} \left\{ T: \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}, \quad \begin{tikzcd} \mathcal{H}_{\pi_1} \arrow[r, "\pi_1"] \arrow[d, "T"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_{\pi_1} \arrow[d, "T"] \\ \mathcal{H}_{\pi_2} \arrow[r, "\pi_2"] & \mathcal{H}_{\pi_2} \end{tikzcd} \quad T \circ \pi_1 = \pi_2 \circ T \right\}$$

Bounded linear

and denote $\mathcal{C}(\pi) = \mathcal{C}(\pi, \pi) = \{ T \in B(\mathcal{H}_\pi) : T \circ \pi = \pi \circ T \}$

closed under taking adjoint.

Next class, we will show some results using above definitions. (可见录播)