五一后开始 Abelian group 很多地方都可以视作 $\mathbb{R}^n$

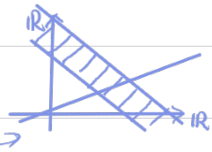
补充 measure的convolution 可被推广到 distribution

$L^1 \subset \mathcal{M}(G)$, $\boxed{\text{convolution of measures}}$

finite Borel measure.

$$\int_{\mathbb{R}^n} f \, d\mu * \nu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x+y)\, d\mu(x)\, d\nu(y)$$

$$\mu \times \nu \xrightarrow{\pi(x,y)=x+y} \pi_*(\mu \times \nu)$$



If μ, ν Radon measure $\not\Rightarrow$ $\mu * \nu$ is also Radon

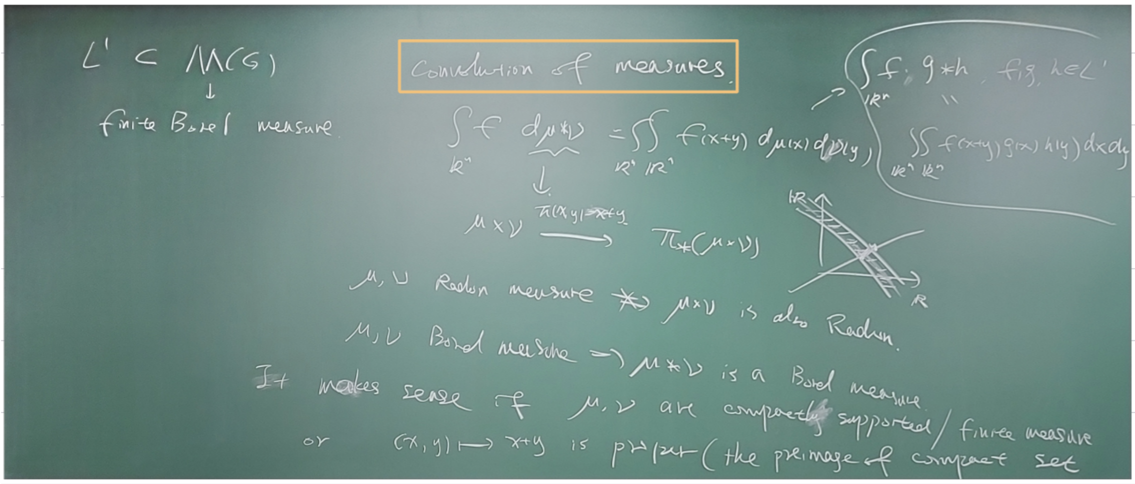
If μ, ν Borel measure $\Longrightarrow$ $\mu * \nu$ is a Borel measure

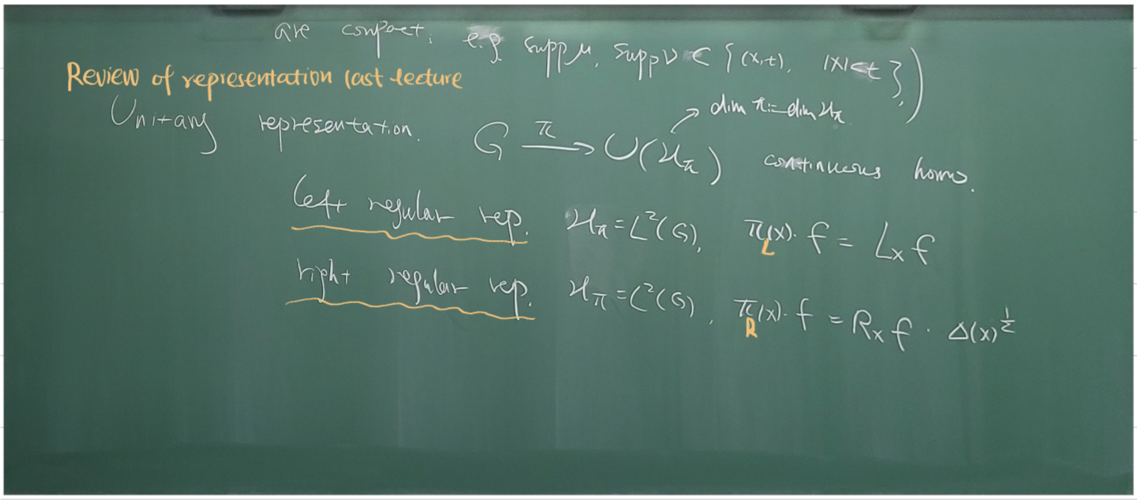
It makes sense if μ, ν are compactly supported / finite measure

or $(x,y) \mapsto (x+y)$ is proper

the pre-image of compact set are compact

e.g. $\operatorname{supp}\mu, \operatorname{supp}\nu \subset \{(x,t),\ |x|<t\}$





Equivalence : $\mathcal{H}_{\pi_1} \xrightarrow{U} \mathcal{H}_{\pi_2}$ U is unitary from $\mathcal{H}_{\pi_1}$ to $\mathcal{H}_{\pi_2}$

$$\begin{array}{ccc} \mathcal{H}_{\pi_1} & \xrightarrow{U} & \mathcal{H}_{\pi_2} \\ \downarrow \pi_1 & \circlearrowright & \downarrow \pi_2 \\ \mathcal{H}_{\pi_1} & \xrightarrow{U} & \mathcal{H}_{\pi_2} \end{array}$$

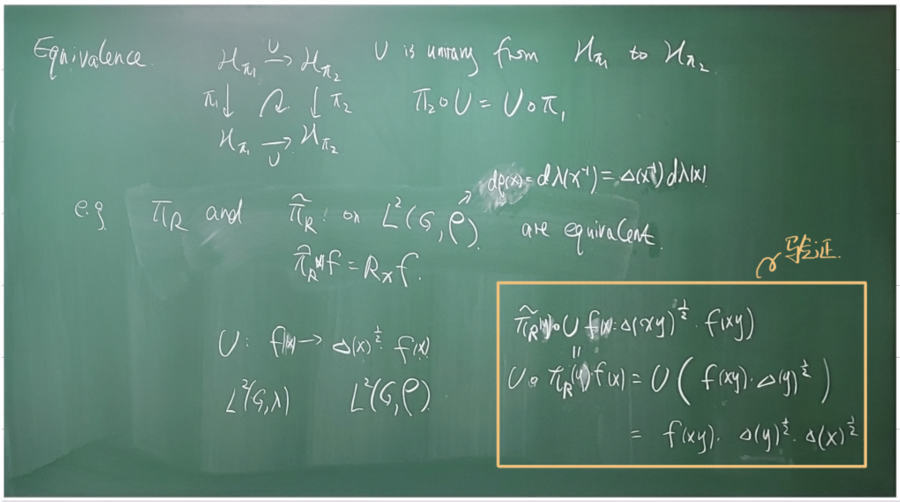
$\pi_2 \circ U = U \circ \pi_1$ (the diagram commutes)

e.g. π_R and $\tilde{\pi}_R$ on $L^2(G, \rho)$ are equivalent

$d\rho(x) = d\lambda(x^{-1}) = \Delta(x^{-1})\, d\lambda(x)$

$$\tilde{\pi}_R(x) f = R_x f$$

$$U : \underset{L^2(G,\lambda)}{f} \to \underset{L^2(G,\rho)}{\Delta(x)^{\frac{1}{2}} f}$$



Def: $\mathcal{C}(\pi_1, \pi_2) \overset{\text{def}}{=} \{ T \in B(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2}),\ T \circ \pi_1 = \pi_2 \circ T,\ \forall x \}$

$\mathcal{C}(\pi, \pi) = \mathcal{C}(\pi)$, $\pi \circ T = T \circ \pi$ (commutant or centralizer of π.)

Def: Say $\mathcal{M}$ is closed subspace of $\mathcal{H}_\pi$, then $\mathcal{M}$ is called invariant if $\pi(x)\mathcal{M} \subset \mathcal{M}$, $\forall x \in G$,

then $\pi|_\mathcal{M}$ is called a subrepresentation, and we call π $\boxed{\text{reducible}}$ if a proper $\mathcal{M}$ exists

otherwise we call it irreducible.

Prop 3.1: If $\mathcal{M}$ is invariant under π, then so is $\mathcal{M}^\perp$ $\Rightarrow$ **Cor**: $\pi = \pi|_\mathcal{M} \oplus \pi|_{\mathcal{M}^\perp} = \pi^\mathcal{M} \oplus \pi^{\mathcal{M}^\perp}$

proof: If $u \in \mathcal{M}$, $v \in \mathcal{M}^\perp$, $\forall x \in G$, $\langle \pi(x) v, u \rangle = \langle v, \pi(x^{-1}) u \rangle = 0$

so $\pi(x) v \in \mathcal{M}^\perp$ ◻

Remark: For non-unitary representation, it may fails.

Say $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \curvearrowright \mathbb{R}^2$, the only invariant space is span$\{(1,0)\}$

Def: π is called cyclic if $\exists\, u \in \mathcal{H}_\pi$ s.t. $\mathcal{H}_\pi = \overline{\{\pi(x)u : x \in G\}}$
$\overset{\text{def}}{=} \mathcal{M}_u \rightarrow$ Invariant

Prop: Every unitary representation is a direct sum of cyclic representation.

proof: by prop 3.1 and Zorn's lemma contradicting the maximality.

Cor: Irreducible representation must be cyclic.

Prop 3.4: $\mathcal{M}$ is a closed subspace of $\mathcal{H}_\pi$, and let P be the orthogonal projection $P: \mathcal{H}_\pi \to \mathcal{M}$

then $\mathcal{M}$ is invariant if and only if $P \in \mathcal{C}(\pi)$ (i.e. $\pi \circ P = P \circ \pi$)

Proof: "$\Leftarrow$": If $\pi\circ P = P\circ\pi$, and $v\in\mathcal{M}$, then $\boxed{\pi(x)v} = \pi(x)Pv = \boxed{P\pi(x)v}\in\mathcal{M}$. So $\mathcal{M}$ is invariant

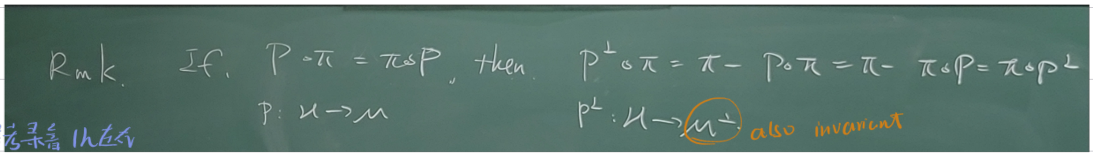
"$\Rightarrow$" If $\mathcal{M}$ is invariant, we have $\pi(x)Pv = \pi(x)v = P\pi(x)v$, for $v\in\mathcal{M}$.

$\mathcal{M}^\perp$ also invariant!

and $\pi(x)Pv = 0 = P\pi(x)v$, for $v\in\mathcal{M}^\perp$ (prop 3.1)

Hence $\pi(x)P = P\pi(x)$. ▨

Remark:
↓
书上没有用号表示 (h在右)



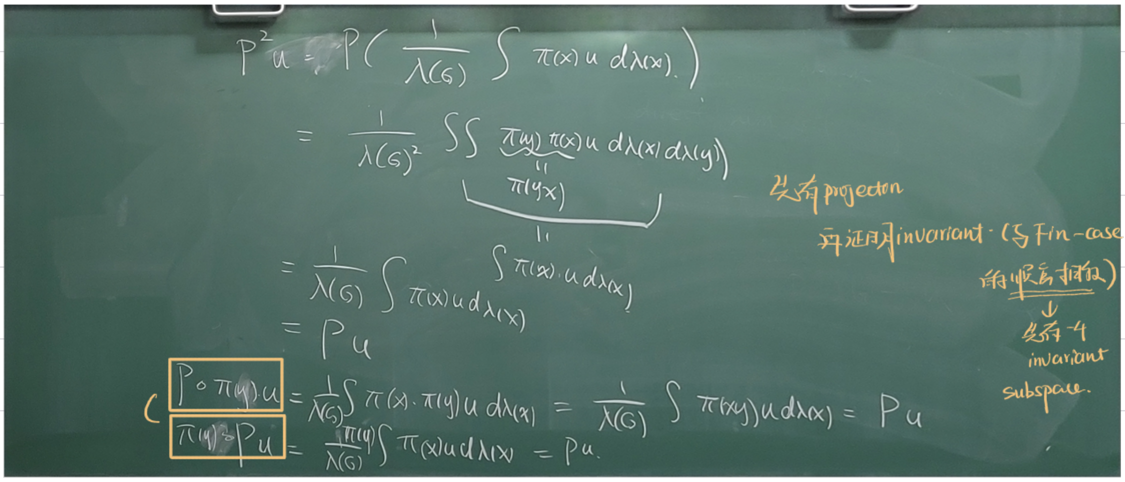
So "$\underbrace{P\circ\pi = \pi\circ P}$" $\Rightarrow$ both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant $\Rightarrow$ $\pi = \pi^{\mathcal{M}}\oplus\pi^{\mathcal{M}^\perp}$

↳ implies thm 3.1

When G is compact (unimodular), one can write $P = \frac{1}{\lambda(G)}\int\pi(x)\,d\lambda(x)$, in particular when G is finite,

then $\boxed{P = \frac{1}{|G|}\sum_{x\in G}\pi(x)}$ 有限群表示论结果

↓
验证 P 为一个 projection



先证这个再把 Schur's lemma 当作 corollary

Theorem: If G is Abelian, then every irreducible unitary representation is one-dimensional ($\mathcal{H}_\pi\cong\mathbb{C}$)

Proof: It suffices to prove $\pi(x) = C_x\cdot I$ (identity on $\mathcal{H}_\pi$)

If not, one of $A \overset{\text{def}}{=} \frac{\pi(x)+\pi^*(x)}{2}$, $B \overset{\text{def}}{=} \frac{\pi(x)-\pi^*(x)}{2}$ is not a multiple of I. Say A (self-adjoint) is not

Since G is abelian, A commutes with all $\underset{\oplus_\lambda\pi_\lambda(y)}{\pi(y)}$, $y\in G$.

As A is self-adjoint, P_λ commutes with all $\pi(y)$, $y\in G$.

↳ projection to eigenspace associated to λ (eigenvalue of A)

↓
则行.

Since G is Abelian, A commutes with all $\pi(y)$, $y\in G$.

As A is self-adjoint, $\underbrace{P_\lambda}_{\text{Proj to eigenspace associated to } \lambda}$ commutes with all $\pi(y)$, $y\in G$ $\quad \bigoplus_\lambda \pi_\lambda(y)$

λ: eigenvalue of A.

$\forall u\in E_\lambda$,

$\lambda\, \pi(y)\cdot u = \pi(y)\cdot Au = A\, \pi(y)u$

$\Rightarrow \pi(y)\cdot E_\lambda \to E_\lambda$.

$\Rightarrow \pi(y) = \bigoplus_\lambda \pi_\lambda(y)$, $\quad \pi_\lambda(y) = \pi(y)\big|_{E_\lambda}$

Then $\forall v\in \mathcal{H}$, $v = \bigoplus_\lambda v_\lambda$, $v_\lambda\in E_\lambda$,

then $\pi(y)\cdot v = \sum_\lambda \pi_\lambda(y)\cdot v_\lambda$

So $\pi(y)\circ P_\lambda v = \pi(y) v_\lambda = \pi_\lambda v_\lambda = P_\lambda\left(\sum_\lambda \pi_\lambda v_\lambda\right) = P_\lambda\, \pi(y)\cdot v$.

$\Rightarrow$ every eigenspace is invariant.

as π is irreducible $\Rightarrow$ $\boxed{\exists!\ \text{eigenvalue}}$ $\Rightarrow$ A is a multiple of I, contradiction. ∎

unitary self-adjoint

the Schur's lemma is more general!

Section 3.3: Functions of positive type $P\subset L^\infty$

Def: ϕ s.t. $\int (f^* * f)\,\phi \geq 0$, $\forall f\in L^1(G)$ $\quad \hat{\mu}$

$\Vert$ In Euclidean space

$\int |\hat{f}|^2 d\mu$

proved in chapter 4

Bochner thm: When G is abelian $P = \{\hat{\mu} : \underbrace{\mu\in M(G)}_{\text{finite Borel measure}}\}$

will be used later

Prop 3.15: $\boxed{\langle \pi(x)u, u\rangle \in P}$ $\forall u$

proof: $\int f^* * f(x)\, \langle \pi(x)u, u\rangle\, dx$

$= \iint \Delta(y^{-1})\, \overline{f(y^{-1})}\, f(y^{-1}x) \langle \pi(x)u, u\rangle\, dy\, dx$

$= \iint \overline{f(y)}\, f(x)\, \langle \pi(y^{-1}x)u, u\rangle\, dy\, dx$ $\quad \pi(y)^*\pi(x)u$ $\quad = \iint \overline{f(y)}\cdot f(x)\, \langle \pi(x)u, \pi(y)u\rangle\, dy\, dx$

$= \iint \langle f(x)\pi(x)u, f(y)\pi(y)u\rangle\, dy\, dx$

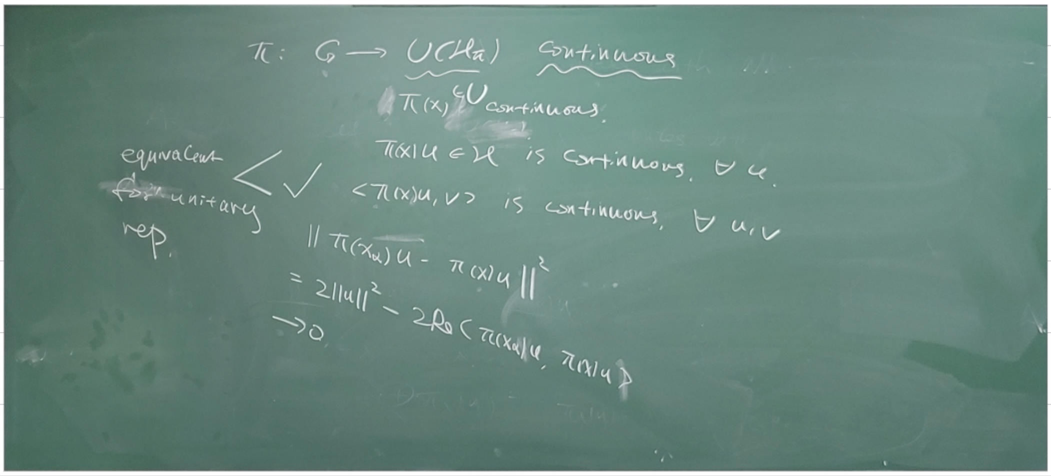
$= \|\pi(f)u\|^2 \geq 0$. ∎

最后考虑一些 topology 相关的问题！

$\pi: G \to U(\mathcal{H}_\pi)$ continuous $\quad$ Strongest

$\pi(x)\in U$ continuous

Equivalent for unitary representation $\begin{cases} \pi(x)u \in \mathcal{H} \text{ is continuous}, \forall u \\ \langle \pi(x)u, v\rangle \text{ is continuous}, \forall u, v\ ✓ \end{cases}$ $\quad$ Weakest


π: G → U(H_π) continuous
π(x) ∈ U continuous.
π(x)u ∈ H is continuous. ∀ u.
<π(x)u, v> is continuous. ∀ u, v
equivalent for unitary rep.
|| π(x_α) u − π(x) u ||²
= 2||u||² − 2Re < π(x_α)u, π(x)u >
→ 0