05-04

刊误(上一次)：距离Final还有四周！

① $\boxed{\text{span}}$

Def: π is called cyclic if $\exists\, u \in \mathcal{H}_\pi$ s.t. $\mathcal{H}_\pi = \overline{\{\pi(x)u : x \in G\}}$ ↗ 这里的norm都不会c ∵ unitary 所以需要span.

$\overset{\text{def}}{=} M_u$ → Invariant

Prop: Every unitary representation is a direct sum of cyclic representation.

proof: by prop 3.1 and Zorn's lemma ↗ contradicting the maximality.

Cor: Irreducible representation must be cyclic.

↓

② For unitary representation: cyclic ⇒ irreducible representation also correct!

Chapter 4: Analysis on locally compact abelian groups.

↗ 通常用加法，但这本书仍然用乘法

unimodular, and $f * g(x) \overset{\text{convolution可交换}}{=} g * f(x) = \int g(y^{-1}x) f(y)\,dy$

$\parallel$

$\int f(y^{-1}x) g(y)\,dy = \int L_y f(x) g(y)\,dy$

Here $L_y f(x) = f(y^{-1}x) \overset{\text{可交换}}{=} f(xy^{-1})$

$R_y f(x) = f(xy) = f(yx)$

4.1: The Dual group

Recall every irreducible unitary representation on G, must be 1-dimensional (Last lecture)

We may identify $\mathcal{H}_\pi \cong \mathbb{C}$, and $\pi(x)\cdot z = \boxed{\xi(x)}\cdot z \in \mathbb{C}$

↓

$\xi : G \to \mathbb{T} \sim S^1$, continuous group homomorphism

called a character.

Denote $\hat{G} = \{\text{all characters of } G\}$ ↗ later we will equip this with topology, also locally cpt abelian gp.

As $\xi(x) = \langle \xi(x)\cdot 1, 1 \rangle$, by prop 3.15 last lecture $\langle \pi(x)u, u \rangle \in P$, $\xi(x)$ are functions of positive type.

↓ 后面用得到

For the reason of symmetry, denote $\xi(x) \overset{\text{def}}{=} \langle x, \xi \rangle$

↑ G 也为 locally cpt abelian gp. $\hat{G}$ 也可讨论！

Then let $\xi(f) = \int_G \langle x, \xi \rangle f(x)\,dx$ ↗ $f \in L^1$ (Recall that $\pi(f)u = \int \pi(x) u f(x)\,dx \in \mathcal{H}_\pi$, using Riesz representation)

$\pi(f) = \int \pi(x) f(x)\,dx \in \mathcal{B}(\mathcal{H}_\pi)$.

Properties: (special case of thm 3.9, $*$-representation)

① $\xi(f * g) = \xi(f)\cdot\xi(g)$ ② $\xi(f^*) = \xi(f)^*$

③ $\xi(x)\cdot\xi(f) = \xi(L_x f)$

proof: ①: $\xi(f * g) = \int_G \int \langle x, \xi \rangle f(y^{-1}x) g(y)\,dx\,dy \xlongequal{\text{unimodular}} \iint \underline{\langle yx, \xi \rangle} f(x) g(y)\,dx\,dy = \xi(f)\cdot\xi(g)$

$= \underline{\langle y, \xi \rangle \langle x, \xi \rangle}$

being gp homomorphism

② : $\xi(f^*) = \int \langle x,\xi\rangle \overline{f(x^{-1})}\,dx = \overline{\int \underbrace{\overline{\langle x^{-1},\xi\rangle}}_{\langle x,\xi\rangle} f(x)\,dx} = \overline{\xi(f)} = \xi(f)^*$

, As in $\boxed{\mathbb{T}}$, $\alpha^{-1} = \bar{\alpha}$, $\underset{1}{\underset{\parallel}{\langle 1,\xi\rangle}} = \langle x,\xi\rangle \cdot \langle x^{-1},\xi\rangle$

then $\langle x,\xi\rangle^{-1} = \langle x^{-1},\xi\rangle = \overline{\langle x,\xi\rangle}$

③ : $\xi(x)\cdot\xi(f) = \int \underbrace{\langle x,\xi\rangle\langle y,\xi\rangle}_{\text{gp homo} = \langle xy,\xi\rangle} f(y)\,dy = \int \langle y,\xi\rangle f(x^{-1}y)\,dy = \xi(L_x f)$.

Hence $\xi(f) = \int_G \langle x,\xi\rangle f(x)\,dx$ defines a non-zero multiplicative functional on $L^1(G)$.

↑ may take $f = \overline{\langle x,\xi\rangle}\big|_{\text{compact}}$.

Now every non-zero multiplicative functional on $L^1(G)$ can be written in the form of the above — 找到 character.

also linear. $\hat{\Phi}$

is given by $\boxed{\text{integration against a character}}$ γ (analogue to Thm 3.11)

proof : $\forall\, \Phi \in L^1(G)^* \cong L^\infty(G)$, then $\exists$ corresponding $\phi \in L^\infty(G)$. $\Phi(f) = \int \phi\cdot f$

then $\forall f, g \in L^1(G)$

$$\int \Phi(f)\,\phi(x) g(x)\,dx = \Phi(f)\Phi(g) = \Phi(f*g)$$

⇓ Abelian

$$= \iint \phi(x)\, f(y) g(y^{-1}x)\,dx\,dy$$

$$= \iint \phi(yx) f(y) g(x)\,dx\,dy = \int \big(\underbrace{\int \phi(yx) f(y)\,dy}_{\int \phi(y) f(x^{-1}y)\,dy \,=\, \Phi(L_x f)}\big)\, g(x)\,dx$$

$$= \int \Phi(L_x f)\, g(x)\,dx$$

Overall $\Phi(f)\phi(x) = \Phi(L_x f) \Rightarrow \boxed{\phi(x) = \frac{\Phi(L_x f)}{\Phi(f)}}$, $\forall f$ s.t. $\Phi(f) \neq 0$

- ϕ being continuous, $\|L_x f - f\|_1 \to 0$, as $x \to 0$
- ϕ being homomorphism. $\phi(xy)\Phi(f) = \Phi(L_{xy} f) = \Phi(L_x L_y f)$

$$= \phi(x)\Phi(L_y f) = \phi(x)\phi(y)\Phi(f)$$

In particular. $\phi(x^n) = \phi(x)^n$, $\forall n \in \mathbb{Z}$, $\phi \in L^\infty \Rightarrow |\phi| = 1$, or 0

If $\phi(x) = 0$ for some x. $\phi(e) = \phi(x)\cdot\phi(x^{-1}) = 0 \Rightarrow \phi(y) = \phi(1)\cdot\phi(y) = 0$, $\forall y$

$\Rightarrow \Phi = 0$, contradiction!

$\Rightarrow \phi$ is a continuous homomorphism from G to $\mathbb{T}$.

▥

In short, $\hat{G}$ can be identified with $\{$non-zero mulitiplicative bounded linear functionals on $L^1(G)\} \subset L^1(G)^* \cong L^\infty(G)$

$\xi(f) = \int \langle x,\xi\rangle f(x)\,dx$ ⇓ Inherits weak-* topology.

From the argument above.

$\hat{G} \cup \{0\} = \{$all multiplicative functionals on $L^1(G)\}$, closed in $L^\infty(G)$. w.r.t weak-* topology.

去除0后,仅需W*Hausdorff (HW)

W*- compact $\Rightarrow \hat{G}$ is W* locally cpt. and contained in the $\boxed{\text{unit ball of } L^\infty}$ weak-cpt (Banach-Alaoglu)

Group structure on $\widehat{G}$, $\langle x, \xi_1\cdot\xi_2\rangle \overset{\text{def}}{=} \xi_1(x)\cdot\xi_2(x)$, $e = 1_G$ constant function.

$$\begin{aligned} \langle x, \xi_1\cdot\xi_2\rangle &\overset{\text{def}}{=} \xi_1(x)\cdot\xi_2(x) \\ &\quad\ \parallel \\ \langle x, \xi_2\cdot\xi_1\rangle &= \xi_2(x)\cdot\xi_1(x) \end{aligned}$$

multiplication and inversion are continuous by dominated convergence thm.

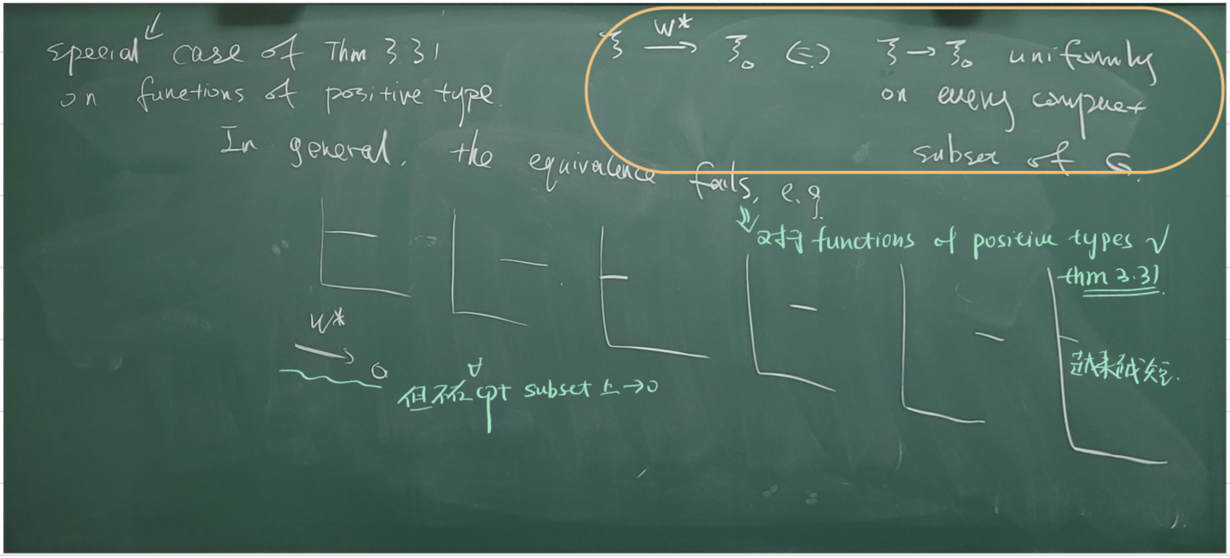
$\Downarrow$

Overall $\widehat{G}$ is a locally compact Abelian group.

More about topology on $\widehat{G}$: the weak-* topology coincides with the compact convergence topology.

$$\xi \xrightarrow{w^*} \xi_0 \Leftrightarrow \xi \to \xi_0 \text{ uniformly on every cpt subset of } G$$

In general, the equivalence fails.



special case of Thm 3.31 on functions of positive type

$\xi \xrightarrow{w^*} \xi_0 \Leftrightarrow \xi \to \xi_0$ uniformly on every compact subset of G.

In general, the equivalence fails, e.g.

$\Downarrow$ 对于 functions of positive types ✓ thm 3.31

但不在∀cpt subset 上→0

就未成立.

$\Leftarrow$

Proof of ⑪: If $\xi \to \xi_0$ uniformly on every compact subset, then $\forall f \in L^1$, $\exists g \in C_c(G)$, $\|f-g\|_{L^1} < \varepsilon$

$$\Rightarrow \left|\int \langle x, \xi-\xi_0\rangle f(x)\,dx\right| \le \Big|\int_{\underset{\text{cpt}}{\operatorname{supp} g}} \underbrace{\langle x, \xi-\xi_0\rangle}_{\to 0 \text{ uniformly}} g(x)\,dx\Big| + 2\|f-g\|_{L^1} \to 0$$

$\Rightarrow$, conversely, $\xi \xrightarrow{w^*} \xi_0$, it sufficies to show that $\exists$ neighborhood U of $e \in G$.

s.t. $\xi \to \xi_0$ uniformly on U.

Let V be a compact neighborhood of e that will be clarified later. denote $f(x) = \frac{1}{|V|}\chi_V(x)$

then $\qquad\qquad$ $|V|$ measure $< \infty$

$$|\xi(x) - \xi_0(x)| \le \underbrace{|\xi(x) - f*\xi(x)|}_{\text{I}} + \underbrace{|f*(\xi-\xi_0)(x)|}_{\text{II}} + \underbrace{|\xi_0(x) - f*\xi_0(x)|}_{\boxed{\text{III}}}$$

$\boxed{\text{III}}$: ξ_0固定 III最简单

$$\text{III} := \left|\frac{1}{|V|}\int_V \xi_0(x) - \underbrace{\xi_0(y^{-1}x)}_{\overline{\xi_0(y)}\,\xi_0(x)}\,dy\right| \le \underbrace{\frac{1}{|V|}\int_V |1 - \overline{\xi_0(y)}|\,dy}_{与x无关} \le \sup_{y\in V} |1 - \overline{\xi_0(y)}| < \varepsilon$$

When V is "small" (independent of x and ξ)

$\Downarrow$

Now fix V. $\qquad$ e.g. V cpt neighborhood $\cap \{y : |1 - \overline{\xi_0(y)}| < \varepsilon\}$

$$\text{now I} \le \left|\frac{1}{|V|}\int_V 1 - \xi(y)\,dy\right| \le \underbrace{\frac{1}{|V|}\int_V |1 - \xi_0(y)|\,dy}_{\le \varepsilon, \text{ by III}} + \underbrace{\frac{1}{|V|}\left|\int_V \xi(y) - \xi_0(y)\,dy\right|}_{\to 0, \text{ as } \xi \xrightarrow{w^*} \xi_0, \text{ independent in } x}$$

It remains to consider II

$\int f(x)\overline{(\xi-\xi_0)(y)}\,dy$ consider conjugate on entire integral.

$$I := \left| \int f(y^{-1}x)\,(\xi-\xi_0)(y)\,dy \right| = \left| \underline{\int \overline{\overline{f(yx)}}\,(\xi-\xi_0)(y)}\,dy \right|$$

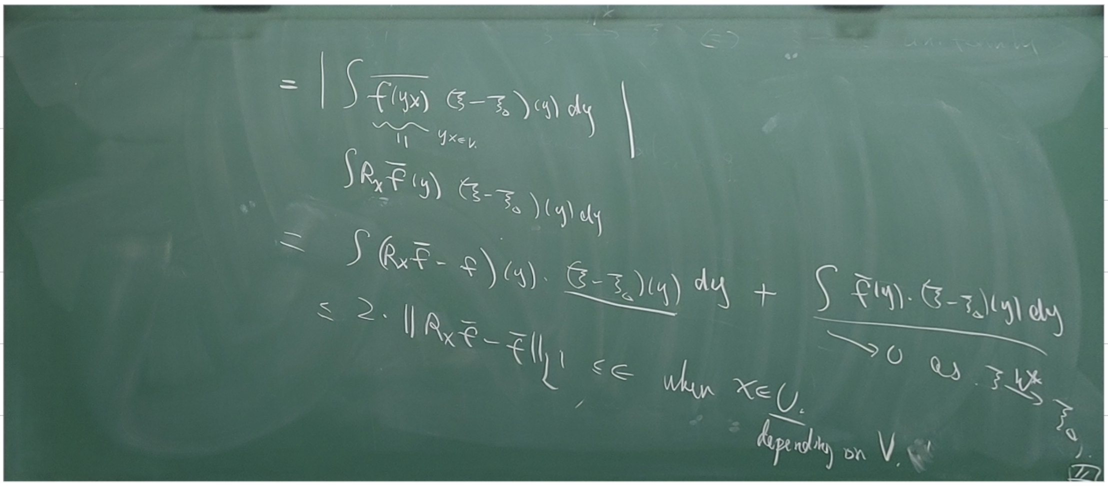
$$\parallel$$

$$\int R_x\overline{f}(y)\,(\xi-\xi_0)(y)\,dy$$

$$= \int (R_x\overline{f} - f)(y)\,(\xi-\xi_0)(y)\,dy + \underbrace{\int \overline{f}(y)\,(\xi-\xi_0)(y)\,dy}_{\to 0,\ \text{as } \xi \xrightarrow{w^*} \xi_0}$$

$$\le 2\cdot\|R_x\overline{f} - \overline{f}\|_{L^1} < \varepsilon,$$ when $x \in U$, depending on V.

▥



$$= \left| \int \underbrace{\overline{f(yx)}}_{\parallel}\,(\xi-\xi_0)(y)\,dy \right| \quad yx \in V.$$

$$\int R_x\overline{f}(y)\,(\xi-\xi_0)(y)\,dy$$

$$= \int (R_x\overline{f} - f)(y)\cdot\underline{(\xi-\xi_0)(y)}\,dy + \underbrace{\int \overline{f}(y)\cdot(\xi-\xi_0)(y)\,dy}_{\to 0 \text{ as } \xi \xrightarrow{w^*} \xi_0}$$

$$\le 2\cdot\|R_x\overline{f} - \overline{f}\|_{L^1} \le \epsilon$$ when $x \in \underline{U}$.

depending on V.