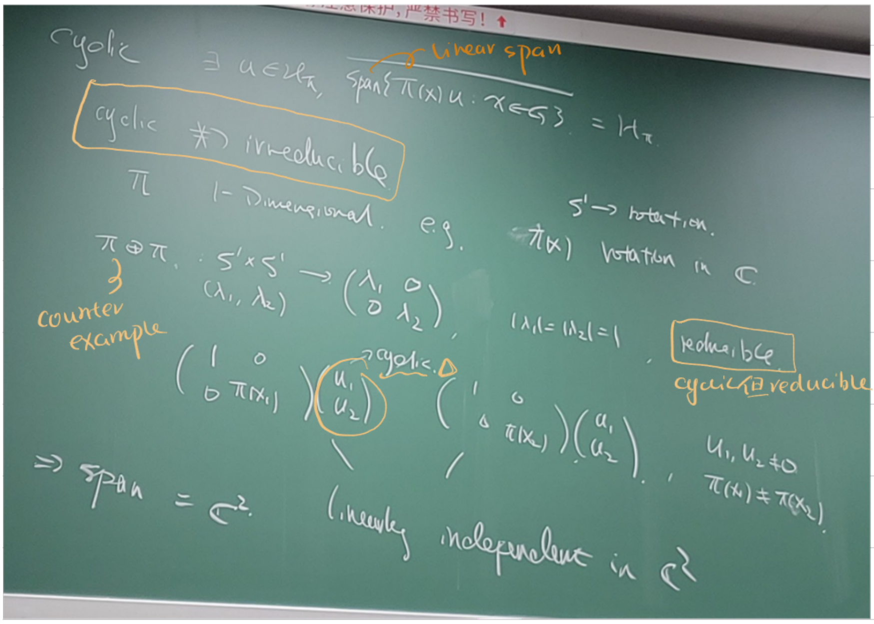
05-09

Summer school：须录音 ← 大概在7月底 3~5天

之前的引误 也有问题



last lecture: $\widehat{G}$: dual group = {characters} $\xi_1\cdot\xi_2(x) = \xi_1(x)\cdot\xi_2(x) = \langle x,\xi_1\rangle\cdot\langle x,\xi_2\rangle$

$\Downarrow$ $\widehat{G}\cup\{0\}$ is compact in $L^\infty \cong (L^1)^*$

Prop 4.5: If G is compact then $\widehat{G}$ is discrete.

需要 prop 4.4 If G is discrete, then $\widehat{G}$ is compact.

Prop 4.4: If G is compact, then $\widehat{G}$ is an orthonormal set in $L^2(G)$ $\widehat{G}\subset L^\infty(G)\subset L^2(G)$

Proof of prop 4.4: Since $\int \xi\cdot\overline{\eta} = \int \langle x,\xi\eta^{-1}\rangle dx$ ($x_0\cdot x_0^{-1}\cdot x$, then by group homomorphism)

$= \langle x_0,\xi\eta^{-1}\rangle \int \langle x_0^{-1}x,\xi\eta^{-1}\rangle dx$ by left-Haar measure

$= \langle x_0,\xi\eta^{-1}\rangle \int \langle x,\xi\eta^{-1}\rangle dx = \langle x_0,\xi\eta^{-1}\rangle \int \xi\cdot\overline{\eta}$

$$\Rightarrow \int \xi\cdot\overline{\eta} = \begin{cases} 1, & \xi=\eta \\ 0, & \text{otherwise} \end{cases}$$ ∎

Now we may see the proof of prop 4.5

Proof of prop 4.5: If G is compact, then $1_G \perp \xi$, $\forall \xi\in\widehat{G}\setminus\{1_G\}$, then ($1_G \in \widehat{G}$)

$\{1_G\} = \overline{\widehat{G}\setminus\{1_G\}}^{\,c}$, then $\{1_G\}$ is open

$\Downarrow$

单点集 $\{e\}$ is open in $\widehat{G}$ $\Rightarrow$ $\widehat{G}$ is discrete.

Now, suppose G is discrete, then $L^1(G)$ has a unit $\delta(x) = \begin{cases} 1, & x=e \\ 0, & \text{otherwise} \end{cases}$

↓由 G being discrete, δ为一个function!

then there is no way for $\xi \xrightarrow{w^*} 0$, $\xi\in\widehat{G}$ $\Leftrightarrow$ $\langle \xi,\delta\rangle \to 0$), but $\langle \xi,\delta\rangle = \delta(\xi)$ $= 1$

$\Rightarrow \{0\}$ is an isolation point in $\hat{G} \cup \{0\}$, and recall that $\hat{G} \cup \{0\}$ is compact in L^∞

$\Rightarrow \hat{G}$ is compact.

Remark ①: $\int f(x)\overline{\langle x,\xi\rangle}\,dx \in C_0(\hat{G})$, $\hat{\delta} = 1 \Rightarrow \hat{G}$ is compact. (Folland, Rudin 的证明，以上为另一个可行证明)

Remark ②: The second half of this proof actually says $\underset{\text{multiplicative identity}}{\delta} \in L^1(G) \Rightarrow \hat{G}$ is compact.

Together with $\boxed{\text{Pontrjagin duality } \hat{\hat{G}} \cong G}$, we can conclude $\delta \in L^1(G)$ iff G is discrete

proof: "$\Leftarrow$" the 2nd half

"$\Rightarrow$" $\delta \in L^1(G) \overset{\text{2nd-half}}{\Longrightarrow} \hat{G}$ is compact $\overset{\text{1st part}}{\Longrightarrow} \hat{\hat{G}}$ is discrete.

$\hat{\hat{G}} \overset{\text{Pontrjagin}}{\cong} G$ $\square$

E.g. (Thm 4.6)

a: $\hat{\mathbb{R}} \cong \mathbb{R}$, $\langle x,\xi\rangle = e^{2\pi i \xi\cdot x}$

b: $\hat{\mathbb{T}} \cong \mathbb{Z}$, $\langle \alpha, n\rangle = \alpha^n$

c: $\hat{\mathbb{Z}} \cong \mathbb{T}$, $\langle n, \alpha\rangle = \alpha^n$

d: $\hat{\mathbb{Z}_K} \cong \mathbb{Z}_K$, $\langle m,n\rangle = e^{2\pi i\, m\cdot n/K}$

proof: (a): $\forall \phi \in \hat{\mathbb{R}}$, $\phi(0)=1$, so $\exists a>0$ s.t. $\int_0^a \phi \overset{\text{def}}{=} A \neq 0$

then $A\cdot\phi(x) = \int_0^a \phi(t)\phi(x)\,dt = \int_0^a \phi(t+x)\,dt$

$$= \underbrace{\int_x^{a+x} \phi(t)\,dt}$$

take derivative $A\phi'(x) = \underset{\phi(a)\cdot\phi(x)}{\underset{\parallel}{\phi(a+x)}} - \phi(x) = (\phi(a)-1)\phi(x) \Rightarrow \phi$ of the exponential form

微分方程.

Since $|\phi|=1$, then $\phi = e^{2\pi i \xi\cdot x}$ for some $\xi \in \mathbb{R}$.

Conversely, every $\xi \in \mathbb{R}$, $e^{2\pi i x\cdot\xi}$ is a character on $\mathbb{R}$.

(b). $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, note that

$$\begin{tikzcd} \mathbb{R} \arrow[d] \arrow[dr, dashed, "e^{2\pi i\xi x}"] & \\ \mathbb{R}/\mathbb{Z} \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{T} \end{tikzcd}$$

提升 $\Rightarrow \xi \in \mathbb{Z}$ (需要周期性)

For any characters $\Rightarrow e^{2\pi i n\cdot x}$

Conversely, every $e^{2\pi i n\cdot x}$, $n \in \mathbb{Z}$ is a character.

(c) $\hat{\mathbb{Z}} \cong \mathbb{T}$, $\forall \phi \in \hat{\mathbb{Z}}$, take $\underbrace{\phi(1) \overset{\text{def}}{=} \overset{\in\,\mathbb{T}\text{ 的 character}}{\alpha}, \text{ then } \phi(-1) = \alpha^{-1}}_{\phi(n) = \alpha^n}$

(d) $\hat{\mathbb{Z}_K} \cong \mathbb{Z}/K\mathbb{Z}$.

$\square$

Prop 4.7: $\widehat{(G_1 \times \cdots \times G_n)} \cong \widehat{G_1} \times \widehat{G_2} \times \cdots \times \widehat{G_n}$, $\{G_i\}$ are locally compact abelian group

Proof: $\xi_i \in \widehat{G_i}$, then $\langle (x_1, \cdots, x_n), (\xi_1, \cdots, \xi_n) \rangle \overset{\text{def}}{=} \prod_{i=1}^{n} \langle x_i, \xi_i \rangle$ defines a character

conversely, every character χ on $G_1 \times \cdots \times G_n$ can be written as

$$\chi(x_1, \cdots, x_n) = \prod_{i=1}^{n} \chi_i(x_i) \text{ where } \chi_i(x) = \chi(e, e, \cdots, \underset{\substack{\uparrow \\ i\text{-th position}}}{x}, e, \cdots) \text{ is a character on } G_i$$

$\chi(x, y) = \chi(e, y) \cdot \chi(x, e)$ $\boxed{///}$

Application: $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$, $\langle x, \xi \rangle = e^{2\pi i x \cdot \xi}$

$\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$, $\langle x, n \rangle = x^n$

$\widehat{\mathbb{Z}^n} \cong \mathbb{T}^n$

therefore $\widehat{G} \cong G$, for any finite abelian group.

Prop 4.9: G_α compact, then $\widehat{\prod_{\alpha \in A} G_\alpha} = \bigoplus_{\alpha \in A} \widehat{G_\alpha}$

$= \{\prod_{\alpha \in A} \xi_\alpha, \xi_\alpha \in \widehat{G_\alpha}$, and $\xi_\alpha = e_\alpha$ for all but finitely many $\alpha\}$

characters

proof: $\forall \xi \in \widehat{\prod G_\alpha}$, let ξ_α be its restriction on G_α, we shall prove that

$\xi_\alpha = e_\alpha$ for all but finitely many α.

Consider $\{x \in \prod G_\alpha : |\langle x, \xi \rangle - 1| < 1\}$ a neighborhood of e

contains $\prod V_\alpha$ (e.g. topology basis), V_α neighborhood of e_α. $V_\alpha = G_\alpha$ for all but finitely many α

$\Rightarrow$ If $V_\alpha = G_\alpha$,

$|\xi_\alpha(G_\alpha) - 1| < 1$ $\Rightarrow \xi_\alpha = e_\alpha$ as $\xi_\alpha(G_\alpha)$ is a subgroup of $\mathbb{T}$

$\quad \shortparallel$

$\xi(\prod x_\beta)$, $x_\beta = \begin{cases} G_\alpha, & \alpha = \beta \\ e_\beta, & \text{otherwise.} \end{cases}$ $\boxed{///}$

Example: $(\mathbb{Z}/_2)^{\mathbb{W}}$ → countable

Explanation 1:

each character ξ on $\mathbb{Z}/_2$ is $\xi(x) = 1$, or $(-1)^x$, $x = 0, 1$

every ξ on $(\mathbb{Z}/_2)^{\mathbb{W}}$, $\xi = \prod_n \xi_n$, ξ_n is trivial for all but finitely many n.

Explanation 2: via dyadic decomposition

$[0, 1)$

$[0, \frac{1}{2})$ $\quad [\frac{1}{2}, 1)$ $\quad$ base-2 decimal expansion

$[0, \frac{1}{4})$ $[\frac{1}{4}, \frac{1}{2})$ $[\frac{1}{2}, \frac{3}{4})$ $[\frac{3}{4}, 1)$

Lebesgue measure $\leftrightarrow$ Haar measure

$\underset{[0,1)}{\sum a_j 2^{-j}} \leftrightarrow (a_j) \in (\mathbb{Z}/_2)^{\mathbb{W}}$

not a group homomorphism.

What about characters?

non-trivial ξ_n becomes n-th Rademacher function r_n that equals

$1, -1, 1, -1, \cdots$ on $[j2^{-n}, (j+1)2^{-n})$, $j = 0, 1, \cdots, 2^n - 1$

and $(\widehat{\mathbb{Z}_2})^{\mathbb{N}} \rightsquigarrow \{\underset{\text{product}}{\prod_{\text{finite}}} r_n\} \overset{\text{def}}{=} \{w_n\}_{n=0}^{\infty}$

$w_0 = 1$, $w_n = r_1^{b_1} r_2^{b_2} \cdots r_k^{b_k}$, $n = \sum_{j=1}^{k} b_j \cdot 2^{j-1}$

$\Rightarrow \{w_n\}_{n=0}^{\infty}$ is an orthonormal family in $L^2([0,1))$. By plancherel, it is an orthonormal basis.

Walsh functions

对应的Haar measure 一致

next section

Example. $\mathbb{Q}_p$, $r = p^m \cdot \frac{a}{b}$, $(a,b) = 1$, $p \nmid a$, $|r|_p = p^{-m}$

$|r_1 + r_2| \leq \max\{|r_1|, |r_2|\}$, $|r_1 \cdot r_2| = |r_1| \cdot |r_2|$, $\mathbb{Q}_p = \overline{\mathbb{Q}}$

and $\mathbb{Q}_p = \{\underbrace{\sum_{j \geq m}^{\infty} c_j p^j}_{\text{series form}}, m \in \mathbb{Z}, c_j = 0, 1, \cdots, p-1\}$

Also $B(1, 0) \overset{\text{def}}{=} \mathbb{Z}_p = \{\sum_{j \geq 0}^{\infty} c_j p^j, c_j = 0, 1, \cdots, p-1\}$

radius center integer ring

open subgroup

$B(p^{-k}, 0) = \{\sum_{j \geq k}^{\infty} c_j p^j, c_j = 0, 1, \cdots, p-1\}$

We first find a character ξ_1 by $\langle x, \xi_1 \rangle = e^{2\pi i x}$

if $x = \sum c_j p^j$, $e^{2\pi i \sum c_j p^j} = e^{2\pi i \sum_{j \leq -1} c_j p^j}$

finite sum

clearly $\langle x + y, \xi_1 \rangle = \langle x, \xi_1 \rangle + \langle y, \xi_1 \rangle$

ξ_1 is a constant on every coset of $\underset{\text{open}}{\mathbb{Z}_p}$. $\Rightarrow$ then ξ_1 is continuous.

For $y \in \mathbb{Q}_p$, let $\langle x, \xi_y \rangle = \langle xy, \xi_1 \rangle$, also a character

We shall prove $\underset{\mathbb{Q}_p}{y} \mapsto \underset{\widehat{\mathbb{Q}_p}}{\xi_y}$ is an isomorphism between topology group $\mathbb{Q}_p$, $\widehat{\mathbb{Q}_p}$

In particular $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$.

Next lecture.