cont. Hilbert Space.

Parseval's identity, $\|f\|^2 = \sum |\langle f, e_n\rangle|^2$

more generally $\langle f, g\rangle = \sum \langle f, e_n\rangle \overline{\langle g, e_n\rangle}$

this gives an isomorphism between $\mathcal{H}$ (Hilbert space) and ℓ^2, in particular, separable Hilbert spaces are all isomorphism.

Bessel inequality: if $\{e_1, e_2, \cdots\}$, an orthonormal sequence, then

$$\sum |\langle f, e_n\rangle|^2 \le \|f\|^2$$

e.g. ① ℓ^2 with natural basis

or $L^2[0,1]$ in other scenes

② Fourier series $L^2[-\pi,\pi]$, with $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\cdot\bar{g}$, with orthonormal basis $e_n(t) = e^{int}$

to see this first orthogonality ✓, It remains to prove the completeness i.e. $\langle f, e_n\rangle = 0\ \forall n \Rightarrow f = 0$

It's easy to see when f is continuous.

$$\|f\|^2 = \langle f, f\rangle = \langle f - \underbrace{P_n}_{\forall \text{ trigonometric series}}, f\rangle$$

$$\le \max_{[-\pi,\pi]} |f - P_n| \cdot \int_{-\pi}^{\pi} f$$

$$\le c\cdot\|f\|\cdot O(1) \quad \text{in } O(1) \qquad \Rightarrow \|f\| = 0$$

For general $f \in L^2[-\pi,\pi]$, let $g(t) = \int_{-\pi}^{t} f(x)\,dx$ ($g(t)$ is continuous! by f being integrable)

Notice $\langle g, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cdot e^{-int}\,dt$

Integration by parts

$$= \frac{-1}{2\pi i n}\left[g(t)e^{-int}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \underbrace{g'(t)}_{f \text{ a.e}} e^{-int}\,dt\right]$$

$= 0$, as $g(-\pi) = g(\pi) = 0$, and $\langle f, e_n\rangle = 0$

$\Rightarrow \langle g - \langle g, e_0\rangle e_0, e_n\rangle = 0$, $\forall n \Rightarrow g$ is a constant

$\Rightarrow f = g' = 0$ a.e. ▥

As a consequence, $f(t) = \sum \underbrace{\langle f, e_n\rangle e_n}_{\text{unique}} \overset{\text{def}}{=} \sum \hat{f}(n) e^{int}$

and $\|f\|^2 = \sum |\hat{f}(n)|^2$

$\quad\ \| $

$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2$

Remark: Carleson (1966) point-wise convergence a.e.

proposition: $\sum_{n\in\mathbb{Z}} |\hat{f}(n+A)|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$, $\forall A \in \mathbb{R}$.

Proof: by definition $\hat{f}(n+A) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{-i(n+A)t}\,dt$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t)e^{-iAt}]e^{-int}\,dt$$

$$\Rightarrow \sum |\hat{f}(n+A)|^2 = \|f(t)\cdot e^{-iAt}\|^2 = \|f\|^2 = \sum |\hat{f}(n)|^2 \quad ▥$$

Example: take $f=1$, then $\hat{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\,dt = \frac{\sin \pi x}{\pi x}$

now let $A = \frac{t}{\pi}$, then

$$\sum \left[\frac{\sin(n\pi+t)}{n\pi+t}\right]^2 = \|f\|^2 = 1$$

$$\Rightarrow \frac{1}{\sin^2(t)} = \sum_{n=-\infty}^{+\infty} \frac{1}{(n\pi+t)^2}, \quad \forall t \neq 0.$$

cont. eg. ③ The Hardy space $H^2 = \left\{ \begin{array}{l} \text{Analytic functions } f = \sum C_n z^n \\ \text{in } \mathbb{D}, \text{ with } \sum |C_n|^2 < \infty \end{array} \right\}$
开圆盘

with $(f,g) = \sum a_n \bar{b}_n$, if $f = \sum a_n z^n$, $g = \sum b_n z^n$

H^2 is a Hilbert space and z^n as orthonormal basis. In fact H^2 is a closed subspace of $L^2[-\pi,\pi]$, $\ell^2 \triangleq H^2$ (Hilbert space isomorphism)

As $\sum_{n=0}^{\infty} C_n z^n \longmapsto \sum_{n=0}^{\infty} C_n e^{int}$

the following example is more interesting
unit disk

④ $A^2 = \left\{ \begin{array}{l} \text{analytic function } f \text{ in } \mathbb{D} \\ \text{with } \iint_{|z|<1} |f(z)|^2 dx dy < \infty \end{array} \right\}$

先求和再积分 先积分再求和.

Question: connection between A^2 and $\mathcal{H}^2$: $\mathcal{H}^2 \subseteq A^2$ (A^2的 norm 比 $\mathcal{H}^2$小)

A non-trivial inclusion!

Recall the most common example of divergent sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$, then consider

$f(z) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} z^n \notin \mathcal{H}^2 \quad \because \sum_{n=0}^{\infty} \frac{1}{(n+1)} = \infty$

then $\|f\|_{A^2}^2 = \iint_{|z|<1} \left|\sum_{n=0}^{\infty} \underbrace{\frac{1}{\sqrt{n+1}}}_{C_n} z^n\right|^2 dx dy$, since integral on unit disc, use the polar coordinate

$$= \int_0^1 \int_0^{2\pi} \underbrace{\left|\sum_{n=0}^{\infty} \frac{r^n e^{in\theta}}{\sqrt{n+1}}\right|^2}_{\text{Fourier series}} d\theta \cdot r dr$$

$\forall r < 1$

$$= \int_0^1 2\pi \cdot \sum_{n=0}^{\infty} \frac{r^{2n+1}}{n+1} dr = \sum_{n=0}^{\infty} \frac{\pi}{(n+1)^2} < \infty.$$

More generally, $\forall f = \sum_{n=0}^{\infty} C_n z^n$, $\|f\|_{A^2}^2 = \pi \cdot \sum_{n=0}^{\infty} \frac{|C_n|^2}{n+1}$.

Question: Find orthonormal basis of A^2

$$= \sum_{n=0}^{\infty} a_n e_n \longrightarrow \sum_{n=0}^{\infty} |a_n|^2$$

then $\boxed{e_n = \sqrt{\frac{n+1}{\pi}}\, z^n}$ √

and $(f,g)_{A^2} = \sum_{n=0}^{\infty} \frac{a_n \cdot \bar{b}_n}{n+1}$, $f = \sum_{n=0}^{\infty} a_n z^n$, $g = \sum_{n=0}^{\infty} b_n z^n$

Ref Exercise 16，以上的另一种证法，这书有几个 exercises

Remark: From $\|f\|_{A^2}^2 = \pi \cdot \sum_{n=0}^{\infty} \frac{|C_n|^2}{n+1}$, it is easy to see A^2 is complete, thus

a Hilbert space, an alternative proof check Ex. 16

check Exercise 2, 4, 12, 13, 16, 17

important tool, rather abstract. 可以求 f表达式 hence the name.

Section 1.4: Reproducing Kernel.

Def: (functional Hilbert space) Hilbert space of functions

Let S be a set, and $\mathcal{H}$ be a Hilbert space whose elements are functions on S. We say $\mathcal{H}$ is a functional Hilbert space if $\forall x \in S$, the map $f \mapsto f(x)$ is bounded from $\mathcal{H}$ to $\mathbb{C}$ (evaluation)

That is $\forall x \in S$, $\exists M_x > 0$ s.t.

$$|f(x)| \le M_x \|f\|, \quad \forall f \in \mathcal{H}$$

e.g. ① $\ell^2 = \{ \text{functions } f \text{ on } \mathbb{Z}, \sum_{n=-\infty}^{+\infty} |f(n)|^2 < \infty \}$

$$\forall n \in \mathbb{Z}, \ |f(n)| \le \|f\| = \left(\sum |f(n)|^2\right)^{\frac{1}{2}} < \infty$$

$\Rightarrow \ell^2$ is a functional Hilbert space ($M_x = 1$, $\forall x \in \mathbb{Z}$)

② $L^2[-\pi,\pi]$ is $\boxed{\text{not}}$ a functional Hilbert space

i.e. evaluation not well-define

($f \mapsto f(x)$ not well-defined, rigor proof later)

Now let $\mathcal{H}$ be a functional Hilbert space i.e. $\forall x \in S$.

$f \mapsto f(x)$ is a bounded linear functional on $\mathcal{H}$

By Riesz representation theorem: $\exists K_x \in \mathcal{H}$ s.t. $f(x) = \langle f, K_x \rangle$

Notice $K_y(x) = \langle K_y, K_x \rangle \overset{\text{def}}{=} K(x,y)$, reproducing kernel of $\mathcal{H}$ or kernel function

$$\overline{K_x(y)} = \overline{\langle K_x, K_y \rangle}$$

Proposition: If $\{e_1, \cdots\}$ is an orthonormal basis, then

$$\boxed{K(x,y) = \sum e_n(x) \overline{e_n(y)}}$$

or def of reproducing kernel in some references

Proof: $K(x,y) = K_y(x) = \sum \langle K_y, e_n \rangle e_n(x) = \sum \overline{e_n(y)} e_n(x)$ ▯

Corollary: $L^2[-\pi,\pi]$ is not a functional Hilbert space

Proof: orthonormal basis e^{int}, $K(t,t) = \sum |e^{int}|^2 = \infty$ ▯

Actually $\mathcal{H}^2$, A^2 are both functional Hilbert space $L^2 \xrightarrow{\text{mod}}$ [illegible] $\boxed{\text{Paley-Weiner space}}$ √ Important later, mentioned in Stein's complex analysis

Example: $\mathcal{H}^2$, $\forall f(z) = \sum c_n z^n \in \mathcal{H}^2$, $\forall \beta \in \mathbb{D}$

$$|f(\beta)| = |\sum c_n \beta^n| \overset{\text{Hölder}}{\le} \underbrace{\left(\sum |c_n|^2\right)^{\frac{1}{2}}}_{\|f\|_{\mathcal{H}^2}} \cdot \underbrace{\left(\sum |\beta|^{2n}\right)^{\frac{1}{2}}}_{M_\beta} \qquad |f(\beta)| \le M_\beta \|f\|_{\mathcal{H}^2}$$

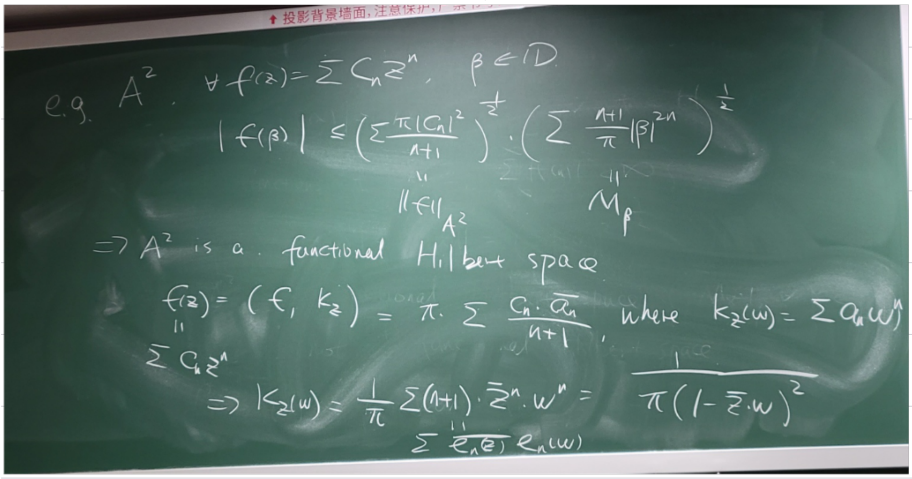
$\Rightarrow \mathcal{H}^2$ is a functional Hilbert space, and $\overset{\sum c_n z^n}{f(z)} = \langle f, K_z \rangle = \sum c_n \overline{a_n}$, if $K_z(w) = \sum a_n \cdot w^n$

$\Rightarrow k_z(w) = \sum (\bar{z})^n w^n = \frac{1}{1-\bar{z}w}$ (Szegö kernel)
$\quad \parallel$
$\sum e_n(w)\overline{e_n(z)}$ $\square$

Example: A^2, $\forall f(z) = \sum c_n z^n$, $\beta \in D$

$$|f(\beta)| \overset{\text{Hölder}}{\leq} \underbrace{\left(\sum \frac{\pi |c_n|^2}{n+1}\right)^{\frac{1}{2}}}_{\|f\|_{A^2}} \cdot \underbrace{\left(\sum \frac{n+1}{\pi} |\beta|^{2n}\right)^{\frac{1}{2}}}_{M_\beta} \Rightarrow |f(\beta)| \leq M_\beta \|f\|_{A^2}$$

$\Rightarrow A^2$ is a functional Hilbert space.



Corollary: $f(z) = (f, k_z) = \frac{1}{\pi}\iint_{|z|\leq 1} \frac{f(w)}{(1-\bar{z}w)^2}\,dx\,dy$ Ref: Exercise 2.3. (ex 4 will be used later)

Question: Is there any way to "modify" $L^2[-\pi,\pi]$ to make it a functional Hilbert space?

Yes!

core of lecture 2

Exercise 4: $\boxed{\text{Paley–Wiener space}}$ PW

$$PW_{[0,1]} = \left\{ f(z) = \int_0^1 \varphi(t) e^{-2\pi i z \cdot t}\,dt,\ \varphi \in L^2[0,1] \right\}$$

or $[-\pi,\pi]$, e.g. plancherel
不用管 2π 系数 v

f is actually an entire function (derivative exists, $\because \varphi \in L^2[0,1]$ compact)
i.e. analytic function over $\mathbb{C}$

With $(f, g) = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\,dx$

Recall plancherel thm: $\int_{\mathbb{R}} |f|^2 = \int_{\mathbb{R}} |\hat{f}|^2$, $\int_{\mathbb{R}} f \cdot \bar{g} = \int_{\mathbb{R}} \hat{f} \cdot \bar{\hat{g}}$, then

$$(f, g) = \int_0^1 \varphi_f \cdot \overline{\varphi_g}, \text{ where } \begin{cases} f(z) = \int_0^1 \varphi_f(t) e^{-2\pi i z \cdot t}\,dt \\ g(z) = \int_0^1 \varphi_g(t) e^{-2\pi i z \cdot t}\,dt \end{cases}$$

thanks to the def, Hölder is available

$\Rightarrow \because L^2[0,1]$ Hilbert, then $PW_{[0,1]}$ is a Hilbert space, for $\forall z$, $|f(z)| \overset{\text{Hölder}}{\leq} \|\varphi\|_{L^2[0,1]} \cdot \underbrace{\|e^{2\pi|\operatorname{Im} z|\cdot t}\|_{L^2[0,1]}}_{M_z}$
$\|f\|_{PW}$

, then $PW_{[0,1]}$ is also a functional Hilbert space.

2 ways to find its reproducing kernel

(1) $f(z) = (f, k_z) = \int_{-\infty}^{+\infty} f(x)\overline{k_z(x)}\,dx$

$\parallel$ by similar observation by plancherel

$\int_0^1 \varphi_f e^{-2\pi i z \cdot t}\,dt = \int_0^1 \varphi(t) \cdot \overline{\varphi_{k_z}(t)}$

$$\Rightarrow k_z(w) = \int_0^1 e^{2\pi i \bar{z} \cdot t} \cdot e^{-2\pi i w \cdot t}\,dt = \frac{e^{2\pi i(\bar{z}-w)} - 1}{2\pi i(\bar{z}-w)}$$

$$\Rightarrow k_z(w) = \int_0^1 e^{2\pi i z t} \cdot e^{-2\pi i w t}\, dt$$

$$= \frac{e^{2\pi i (z-w)} - 1}{2\pi i (z-w)}$$

(2) From $(f,g) = \int_0^1 \varphi_f \cdot \overline{\varphi_g}$, one can conclude that

$$\int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i z t}\, dt = \widehat{e^{2\pi i z t}}(n)$$ is an orthonormal basis.

Fourier transform

$$\Rightarrow K(w,z) = \sum e_n(w)\,\overline{e_n(z)}$$

$$= \sum \overline{\widehat{e^{2\pi i w t}}(n)} \cdot \widehat{e^{2\pi i z t}}(n)$$

Fourier 级数

对称

$$= \left(e^{2\pi i z t}, e^{2\pi i w t}\right)_{L^2[0,1]} = \text{above}$$