Recall last time $\widehat{\prod_\alpha G_\alpha} = \bigoplus_\alpha \widehat{G_\alpha}$

product　direct sum　不一样

若 G_α 为compact, 则 $\prod_\alpha G_\alpha$ cpt, 且 $\widehat{\prod_\alpha G_\alpha}$ 为discrete, 因此右端需

要为 direct-sum, 不然 不能保证单点集为开集 (discrete)

cont. $\mathbb{Q}_p = \{\sum_{j\geq m} c_j p^j,\ m\in\mathbb{Z},\ c_j = 0,1,\cdots,p-1\}$

let $\xi_1(x) \overset{\text{def}}{=} e^{2\pi i \sum_{j=N_1}^{-1} p^j}$, if $x = \sum_{j\geq N_1} c_j p^j$, $\xi_y(x) \overset{\text{def}}{=} e^{2\pi i y x}$

$\| \quad e^{2\pi i x}$ $\qquad\qquad \langle x, \xi_y\rangle = \langle xy, \xi\rangle$

$\boxed{\text{Goal:}}$ $y \mapsto \xi_y$ is isomorphism

$$\begin{array}{ccc} y & \mapsto & \xi_y \\ \cap & & \cap \\ \mathbb{Q}_p & \cong & \widehat{\mathbb{Q}_p} \end{array}$$

Lemma 4.10: If $\xi \in \widehat{\mathbb{Q}_p}$, $\exists K\in\mathbb{Z}$, s.t. $\xi = 1$ on $B(p^{-K}, 0)$

proof: $\exists K$ s.t. $|\xi(x) - 1| < 1$, $\forall x \in \underbrace{B(p^{-K}, 0)}_{\text{open subgroup}}$

$\Rightarrow \xi(B(p^{-K}, 0))$ is a subgroup $\Rightarrow \xi = 1$ on $B(p^{-K}, 0)$ ▨

Remark: ① $\forall \xi$, $\exists j_0$ s.t. $\xi(p^j) = 1$, $\forall j \geq j_0$, $\xi(p^{j_0 - 1}) \neq 1$

② $\Rightarrow \xi$ is a constant on every ball of radius p^{-K}, so ξ is determined by its value on p^j, $j\in\mathbb{Z}$

$\xi(\sum_{j\geq m} c_j p^j) = \prod_{j=M}^{K-1} \xi(p^j)^{c_j}$

We start from $j_0 = 0$.

Lemma 4.11: Suppose $\overset{w_0}{\langle 1, \xi\rangle} = 1$, $\overset{w_1}{\langle p^{-1}, \xi\rangle} \neq 1$, then

$$\langle p^{-K}, \xi\rangle = e^{2\pi i \sum_{j=1}^{K} c_{K-j} p^{-j}} \quad \forall\, 1,2,3,\cdots, \text{ for some } c_j \in \{0,1,\cdots,p-1\}\ j = 1,2,\cdots,\ c_0 \neq 0$$

proof: Denote $w_K = \langle p^{-K}, \xi\rangle$, then

$$w_K = \langle p^{-K}, \xi\rangle = \langle p\cdot p^{-K-1}, \xi\rangle = \langle p^{-K-1}, \xi\rangle^p = w_{K+1}^p$$

$1 = w_0 = w_1^p \Rightarrow w_1 = e^{2\pi i c_0/p}$, $c_0 \neq 0$

$w_1 = w_2^p \Rightarrow w_2 = e^{2\pi i (\frac{c_0}{p^2} + \frac{c_1}{p})}$, $c_1 \in \{0,1,\cdots,p-1\}$ $\qquad (c_0 + c_1 p)\cdot p^{-2}$

$w_2 = w_3^p \Rightarrow w_3 = e^{2\pi i (\frac{c_0}{p^3} + \frac{c_1}{p^2} + \frac{c_2}{p})}$ $\qquad c_2 \in \{0,1,\cdots,p-1\}$

$(c_0 + c_1 p + c_2)\cdot p^{-3}$

$\cdots$ done by induction! ▨

Now,

Lemma 4.12: $\xi \in \widehat{\mathbb{Q}_p}$, $\langle 1, \xi\rangle = 1$, $\langle p^{-1}, \xi\rangle \neq 1$, then $\exists y \in \underset{|y|=1}{\mathbb{Q}_p}$ s.t. $\xi = \xi_y$.

proof: Take $y = c_0 + c_1 p + c_2 p^2 + \cdots$, $|y| = 1$, and

$$\langle p^{-K}, \xi\rangle = e^{2\pi i (c_0 + c_1 p + \cdots + c_K p^K)\cdot p^{-K}} = \langle y\cdot p^{-K}, \xi_1\rangle = \langle p^{-K}, \xi_y\rangle \Rightarrow \xi = \xi_y.$$ ▨

Thm 4.13: $y \mapsto \xi_y$ is an isomorphism between $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}_p}$

proof: group homomorphism ✓

Injective ✓ $\langle x, \xi_y \rangle = e^{2\pi i x \cdot y}$ ✓

Now we show that it is surjective: $\forall \xi \in \widehat{\mathbb{Q}_p}$, $\exists$ smallest integer j s.t.

$\langle p^j, \xi \rangle = 1$. then consider η, s.t. $\underbrace{\langle x, \eta \rangle = \langle p^j x, \xi \rangle}_{\text{character}}$

By previous lemma, $\eta = \xi_y$, for some $y \in \mathbb{Q}_p$, $|y| = 1$

$\Rightarrow \langle x, \xi \rangle = \langle p^j p^{-j} x, \xi \rangle = \langle p^{-j} x, \eta \rangle = \langle p^{-j} x, \xi_y \rangle = \langle x, \xi_{p^{-j} y} \rangle$

$\Rightarrow \xi = \xi_{p^{-j} y}$

Hence $y \mapsto \xi_y$ is a group isomorphism. It remains to show that it is homeomorphism. topology

Consider the neighborhood base of $1 \in \widehat{\mathbb{Q}_p}$

weak* topo

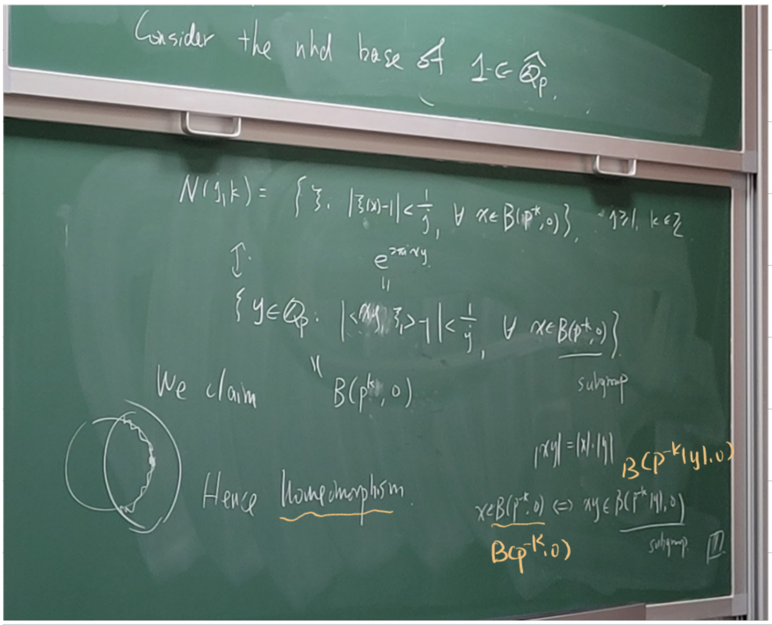
$N(j,k) = \{\xi : |\xi(x) - 1| < \frac{1}{j},\ \forall x \in B(p^{-k}, 0)\},\ j \geq 1,\ k \in \mathbb{Z}$

topology basis之对应 $\updownarrow$

$\{y \in \mathbb{Q}_p : |\overset{e^{2\pi i x y}}{\overset{\shortparallel}{\langle xy, \xi_1 \rangle}} - 1| < \frac{1}{j},\ \forall x \in \underbrace{B(p^{-k}, 0)}_{\text{subgroup}}\}$

$\shortparallel$

$B(p^k, 0)$

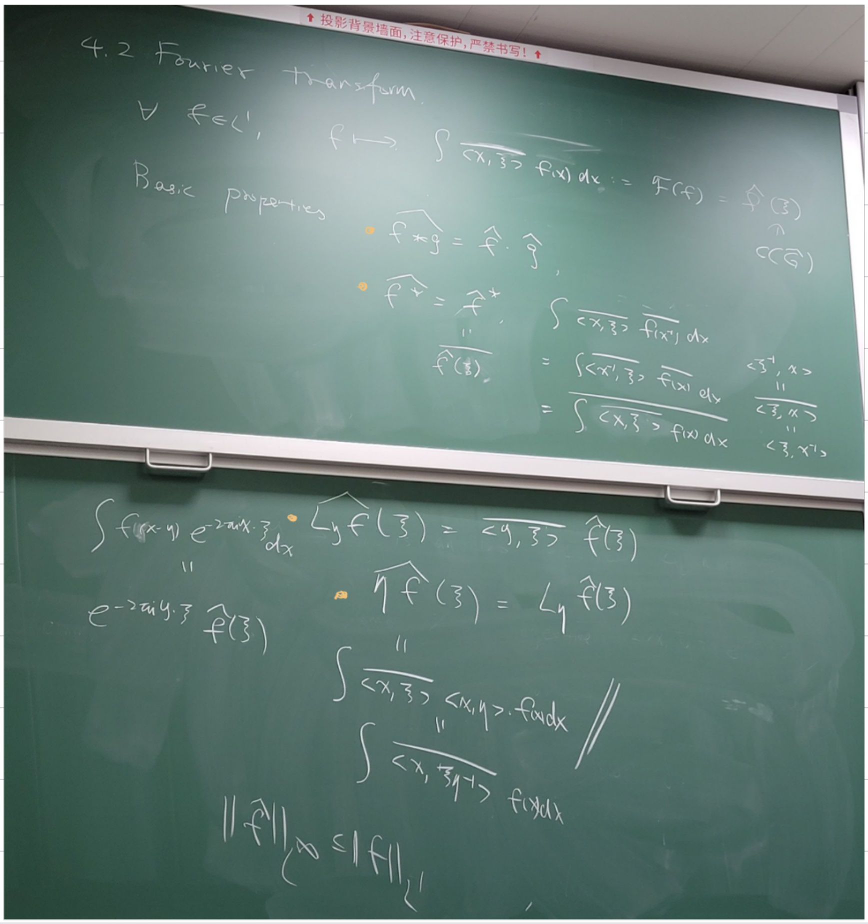


Section 4.2: Fourier Transform

$\forall f \in L^1,\ f \mapsto \int \overline{\langle x, \xi \rangle} f(x)\, dx \overset{\text{def}}{=} \mathcal{F}(f) = \hat{f}(\xi) \in C(\hat{G})$

Basic properties:

- $\widehat{f * g} = \hat{f} \cdot \hat{g}$
- $\widehat{f^*} = \hat{f}^*$
  $\shortparallel$
  $\overline{\hat{f}(\xi)}$
- $\widehat{L_y f}(\xi) = \overline{\langle y, \xi \rangle} \hat{f}(\xi)$



Prop 4.18: $\hat f \in C_0(\hat G)$, and $\mathcal{F}(L^1(G))$ is a dense subset of $C_0(\hat G)$

by stone-weierstrass

proof: $\underbrace{\int \overline{\langle x,\xi\rangle} f(x)dx}_{\text{continuous in } \hat G \cup\{0\}}$ is continuous in $\xi \in L^\infty$ under weak$*$-topology

Recall that $\hat G \cup \{0\}$ is compact in L^∞ under w-$*$ topology

then $\int \cdots f(x)dx \to 0$, $\Rightarrow \hat f(\xi) \in C_0(\hat G)$ ▯

之前的证法

Recall in $\mathbb{R}^n$, $\forall f \in L^1$, $\exists g \in C_0^\infty$ s.t. $\|f-g\|_{L^1} < \varepsilon$

$$\hat f = \underset{C_0}{\hat g} + \underbrace{\int (f-g) e^{2\pi i}}_{<\varepsilon}$$

More generally, one can define the Fourier transform on $M(G)$, finite complex Radon measure μ.

$$\hat\mu(\xi) = \int \overline{\langle x,\xi\rangle}\, d\mu(x) \in C(\hat G)$$

then

$$\widehat{\mu * \nu} = \hat\mu \cdot \hat\nu, \quad \int \underset{\text{bounded continuous}}{\phi}\, d(\mu*\nu) = \iint \phi(xy)\, d\mu(x)\, d\nu(y)$$

$$\|\hat\mu\|_{L^\infty} \le \|\mu\|_{M(G)}$$

Similarly. $\forall \mu \in M(\hat G)$, one can define $\phi_\mu(x) = \int \langle x,\xi\rangle\, d\mu(\xi) \in C(G)$

Prop 4.18: the map $\mu \to \phi_\mu$ is injective from $M(\hat G)$ to $C(G)$

proof: If $\phi_\mu = 0$, $0 = \int f \bar\phi_\mu = \iint \underset{\in L^1(G)}{f(x)} \overline{\langle x,\xi\rangle}\, d\mu(\xi)\, dx = \int \underbrace{\hat f(\xi^{-1})}_{\text{dense in } C_0(\hat G)} d\mu(\xi)$, for $\forall f \in L^1(G)$

then $\mu = 0$ ▯

If $\mu \in M(\hat{G})$ is positive, then ϕ_μ is a function of positive type

i.e. $\int (f * f^*(x)) \phi_\mu(x)\, dx \geq 0$

$$= \iint f * f^*(x) \, \overline{\langle x, \xi^{-1} \rangle} \, dx \, d\mu(\xi) = \int |\hat{f}(\xi^{-1})|^2 d\mu(\xi) \geq 0$$

Thm 4.19 (Bochner's thm)

If $\phi \in P(G)$, $\exists! \, \mu \in M(\hat{G})$ positive s.t. $\phi = \phi_\mu$.

• In the proof we need $\|\underbrace{f * \cdots * f}_{n}\|_{L^1}^{\frac{1}{n}} \to \|\hat{f}\|_{L^\infty}$, $\forall f \in L^1$

In fact, it can be extended to $\mu \in M(G)$.

Recall that: $\sigma(x) \overset{def}{=} \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}$

$R(\lambda) = (\lambda e - x)^{-1}$, analytic in $\lambda \in \mathbb{C} \setminus \sigma(x)$

$\rho(x) \overset{def}{=} \sup\{|\lambda| : \lambda \in \sigma(x)\} \leq \|x\|$

谱半径

Thm 1.8: $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$

proof: $\lambda^n e - x^n = (\lambda e - x) \cdot \sum_0^{n-1} \lambda^j x^{n-1-j}$

$= \sum_0^{n-1} \lambda^j x^{n-1-j} (\lambda e - x)$

So $\lambda^n e - x^n$ is invertible $\Rightarrow \lambda e - x$ is invertible.

So $\lambda \in \sigma(x) \Rightarrow \lambda^n \in \sigma(x^n)$, then $|\lambda^n| \leq \|x^n\|$

$\Rightarrow \|x^n\|^{\frac{1}{n}} \geq |\lambda|$, $\forall \lambda \in \sigma(x)$.

$\Rightarrow \liminf \|x^n\|^{\frac{1}{n}} \geq \rho(x)$.

Conversely, $\forall$ bounded linear functional ϕ, $\phi \circ R(\lambda)$ is analytic in $\lambda \in \mathbb{C} \setminus \sigma(x)$

In particular, it is analytic in $\lambda > \rho(x)$.

Recall, when $|\lambda| > \|x\|$

$\lambda e - x = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n \Rightarrow \underbrace{\phi \circ R(\lambda)}_{\text{analytic in } |\lambda| > \rho(x)} = \sum_{n=0}^{\infty} \lambda^{-n-1} \phi(x^n)$, $\forall |\lambda| \geq \|x\|$

$\Downarrow$ by complex analysis

absolutely convergent in $|\lambda| > \rho(x)$

$\Rightarrow |\phi(x^n)| \leq C_\phi \cdot |\lambda|^{n+1}$

Now, by Banach-Steinhaus