05/20

Last time $\|\underbrace{f * \cdots * f}_{n\text{-fold}}\|_{L^1}^{\frac{1}{n}} \to \|\hat{f}\|_{L^\infty}, \ \forall f \in L^1$

this result is also carried in more abstract Banach algebra. (may require being Abelian)

Here we only consider locally compact abelian group

proving the above result requires spectrum radius theorem than

$L^1(G) \qquad \|x^n\|^{\frac{1}{n}} \to \rho(x)$ spectrum radius

We shall prove that $f \in L^1$ $\boxed{\hookrightarrow M(G)}$ then we have identity

Finite complex Radon measure

Now we want to show that $\boxed{\mu \text{ has an inverse in } M(G) \iff \hat{\mu} \text{ has no zero.}}$ what we need to show now.

If so, then $\rho(\mu) = \|\hat{\mu}\|_{L^\infty}$

考 $\lambda E - \mu$ 可不可逆 对应 dirac-delta

Recall that μ has an inverse ν: $\mu * \nu = \delta$, then $\hat{\mu}\hat{\nu} = 1$, now support $\hat{\mu}$ has a zero, then $\hat{\mu}\hat{\nu} \neq 1$, then μ is not invertible! therefore we have shown the $\Rightarrow$ side.

Now it suffices to show "$\Leftarrow$", that is 否命题 if μ is not invertible, then $\hat{\mu}$ has a zero

这部分在书上是之前 general theory 的一部分！

$\exists$ a characters $\xi \in \hat{G}$ s.t. $\int \overline{\langle x, \xi \rangle}\, d\mu(x) = 0$, $\overline{\langle x, \xi\rangle} = \overline{\xi(x)}$

μ 自己生成个 ideal，但这 ideal 不包含 identity (proper ideal) (∵不 invertible)

If μ is not invertible then μ is contained in a maximal ideal J. (by Zorn's lemma)

and the maximal ideal J is non-trivial. Notice that $J \subset \{\text{all non-invertible elements}\}$

Also an ideal. and a closed subset ($\uparrow$ As invertible $\to$ open subset), proper as it does not contain identity.

$\Rightarrow \mu \in$ closed maximal ideal J

consider mod

now we consider $\boxed{M(G)/J}$ closed, with quotient norm

Still a Banach algebra, since $\|\hat{f}\| = \min_{g \in J} \|f + g\|$

J is maximal. We claim that $M(G)/J$ is one-dimensional (Field)

by Former lecture

therefore it is isomorphic to $\mathbb{C}$, i.e. $M(G)/J \cong \mathbb{C}$

Now we have a natural map $\pi: M(G)/J \to \mathbb{C}$, where $\ker(\pi) = J$

can be lifted

$$\begin{array}{ccc} M(G) & \dashrightarrow & \boxed{\tilde{\pi}} \\ p \downarrow & & \\ M(G)/J & \xrightarrow{\pi} & \mathbb{C} \end{array}$$

$\forall$ multiplicative bounded linear functional

therefore $\exists$ a character $\xi \in \hat{G}$, s.t. $\tilde{\pi}(f) = \int \langle x, \xi \rangle f(x)\, dx$, $\forall f \in L^1(G)$

Note that $L^1(G)$ is dense in $M(G)$, we could have $\tilde{\pi}(\nu) = \int \langle x, \xi \rangle\, d\nu(x)$ (Extended to $M(G)$)

for $\forall \nu \in M(G)$

As $\ker(\tilde{\pi}) = J \ni \mu \Rightarrow 0 = \tilde{\pi}(\mu) = \int \langle x, \xi \rangle\, d\mu(x) = \hat{\mu}(\xi)$

$\boxed{\hat{\mu} \text{ has a zero.}}$

Overall, the zero of $\hat{\mu}$ corresponds to those maximal ideals

e.g. Given $\xi \in \hat{G}$, then $\{\mu \in M(G),\ \hat{\mu}(\xi) = 0\} \Leftrightarrow$ maximal ideal $\boxed{//}$

The above result is for the proof of the Bochner's theorem

Thm 4.19 (Bochner)

Functions of positive type $\int (f^* * f)\cdot\phi \geq 0 = \int |\hat{f}(\xi)|^2 d\mu(\xi)$ if $\phi = \phi_\mu$ (we have shown the reverse of Bochner's thm)

If $\phi \in P(G)$, $\exists!\ \mu \in M(\hat{G})$, s.t. $\phi = \phi_\mu$ a.e.

$\int \langle x, \xi\rangle\, d\mu(x)$ 某个正测度的 Fourier transform

proof: Notice that if we have ψ_u is an approximating identity so is $\underbrace{\psi_u^* * \psi_u}_{\text{好与 positive type 联系}}$ is also an approximating identity.

Consider Hermitian form $(f, g)_\phi = \int \phi\, (g^* * f)$

By Cauchy-Schwarz $|(f, g)_\phi|^2 \leq |(f, f)_\phi| \cdot |(g, g)_\phi|$

$$\left|\int \phi\, (g^* * f)\right|^2 \leq \left(\int \phi\, (f^* * f)\right)\left(\int \phi\, (g^* * g)\right)$$

let $g = \psi_u$, and we may assume $\underbrace{\int \phi\, (g^* * g)}_{\text{approximating}} \to C$ (in fact $\phi(0)$)

$\Rightarrow$ take limit on both sides of the inequality

$$\left|\int \phi f\right|^2 \leq C \left(\int \phi\, (f^* * f)\right) \quad \forall f \in L^1$$

$$\Downarrow$$

$$\boxed{\left|\int \phi f\right|} \leq C^{\frac{1}{2}} \Big(\int \phi\, \underbrace{(f^* * f)}_{\text{consider this } f}\Big)^{\frac{1}{2}}$$

$$\leq C^{\frac{1}{2} + \frac{1}{4}} \left(\int \phi\, (f^* * f * f^* * f)\right)^{\frac{1}{4}}$$

Denote $h = f^* * f$, then

$$\leq C \cdot \lim_{n \to \infty} \left|\int \phi\, h^{(n)}\right|^{\frac{1}{2n}} \quad (L^\infty \neq 0)$$

$$\leq C \cdot \lim_{n \to \infty} \|h^{(n)}\|_{L^1}^{\frac{1}{2n}} = C \cdot \|\hat{h}\|_{L^\infty}^{\frac{1}{2}} \text{ by previous result}$$

$$= \boxed{C \cdot \|\hat{f}\|_{L^\infty}}$$

$$\Downarrow$$

Recall $\underset{L^1(G)}{f} \mapsto \hat{f} \in C_0(\hat{G})$ is injective (4.18)

and $\{\hat{f} : f \in L^1\}$ is dense in $C_0(\hat{G})$

$\Rightarrow$ ϕ defines a bounded linear functionals on $C_0(\hat{G})$ (可以 extend)

$$\Downarrow$$

complex finite Radon measure

$$\Downarrow$$

$\exists\ \mu \in M(\hat{G})$ s.t. $\int \phi f = \int \hat{f}(\xi)\, d\mu$

$$= \int \int \overline{\langle x, \xi\rangle} f(x)\, dx\, d\mu(\xi)$$

$$= \int f(x)\, \hat{\mu}(x)\, dx$$

$\Rightarrow \phi = \hat{\mu}$ $\boxed{//}$

Now we will spend a lot of time proving the Fourier inversion.

We first prove some partial results in Fourier Inversion $\Longrightarrow$ $\boxed{\text{Pontrjagin Duality}}$ $\Rightarrow$ Full Form of Fourier Inversion

The most familiar form of Fourier inversion ($f \in L^1$, $\hat f \in L^1 \Rightarrow \hat{\hat f}(x) = f(x^{-1})$, a.e.) ep

空间 matters

We shall prove another version first:

Denote $B(G) = \{\phi_\mu, \mu \in M(\hat G)\}$ $\overset{\text{Bochner}}{=}$ linear span of $P(G)$, where $\phi_\mu = \int \langle x, \xi\rangle d\mu(\xi)$

$B^1(G) = B(G) \cap L^1(G)$

As $\mu \in M(\hat G) \to \phi_\mu \in B(G)$ is a Bijection, Denote its inverse by $\phi \mapsto \mu_\phi$

i.e. $\phi(x) = \int \langle x, \xi\rangle d\mu_\phi(\xi)$ ∃ such μ_ϕ

Thm 4.22 (Fourier Inversion theorem 1)

If $f \in B^1$, then $\hat f \in L^1(\hat G)$, and $f(x) = \int \langle x, \xi\rangle \underbrace{\hat f(\xi)}_{\mu \text{ 就是 } \hat f} d\xi$ if the Haar measure is suitably normalized.

$\Updownarrow$

$d\mu_f(\xi) = \hat f(\xi) d\xi$. (a sense of Fourier inversion)

Lemma 4.20: If $K \subset \hat G$ is compact, then $\exists f \in C_c(G) \cap P$ (function of positive types)

s.t. $\hat f \geq 0$ on $\hat G$, and $\hat f > 0$ on G

Proof: We only need to consider open neighborhood (then by finite covering (K compact))

Pick $h \in C_c(G)$, with $\hat h(1) = \int h = 1$, and set $g = h^* * h$ (Will found transform, also g is a function of positive type.)

then g is also in $C_c(G)$, $\hat g(\xi) = |\hat h(\xi)|^2 \geq 0$

and $\hat g|_V > 0$, for some open neighborhood V of e (identity)

then $\exists \xi_1, \cdots, \xi_n$, s.t. $K \subset \bigcup_{i=1}^n \xi_i^{-1} V$ (Finite covering), and take $f = (\sum_{j=1}^n \xi_j) g$

$\hat f$ 依 $\xi_j\, g$ $\Rightarrow$ $\hat g(\cdot - \xi_j)$, 对于 $\xi_j^{-1} V$ ▥

Lemma 4.21: If $f, g \in B^1$, then $\hat f d\mu_g = \hat g d\mu_f$

Proof: Recall that $\phi(x) = \int \langle x, \xi\rangle d\mu_\phi(\xi)$

$\forall h \in L^1(G)$, then $\int \hat h \, d\mu_f \overset{\text{by def}}{=} \iint \langle x^{-1}, \xi\rangle h(x) dx \, d\mu_f(\xi)$

$= \int f(x^{-1}) h(x) dx$

$= f * h(1)$, Now we use properties of convolution.

therefore $\int \hat g \cdot \hat h \, d\mu_f$

$= \int \widehat{g * h} \, d\mu_f = g * h * f(1)$

$= f * h * g(1) = \int \hat f \hat h \, d\mu_g$

As $\{\hat h : h \in L^1(G)\}$ is dense in $C_0(\hat G)$, we have $\hat g \, d\mu_f = \hat f \, d\mu_g$ ▥

With Lemma 4.20 and Lemma 4.21, we can now prove thm 4.22

Proof of thm 4.22: We need to use the uniqueness of Haar measure $d\xi$.

$\forall \psi \in C_c(\hat{G})$, by lemma 4.20, $\exists\, g \in U$, s.t. $\hat{g} > 0$ on supp ψ, then define

functional

$$I(\psi) = \int \frac{\psi}{\underbrace{\hat{g}}_{>0}}\, d\mu_g$$

, now we want to show I is irrelevant of g using lemma 4.21

Now we show $I(\psi)$ is independent of g (only depends on ψ)

$\forall f$, $\hat{f} > 0$ on supp ψ

$$I(\psi) = \int \frac{\psi}{\hat{g}}\, d\mu_g = \int \frac{\psi}{\hat{f}\hat{g}} \boxed{\hat{f}\, d\mu_g} \overset{\text{lemma 4.21}}{=} \int \frac{\psi}{\hat{f}\hat{g}} \boxed{\hat{g}\, d\mu_f} = \int \frac{\psi}{\hat{f}}\, d\mu_f$$

$\downarrow$

bounded linear functional on $C_c(\hat{G})$

Next, we show left invariance of I, i.e. $\boxed{I(L_\eta \psi) = I(\psi)}$, $\forall \eta \in \hat{G}$

By def, now $\int \langle x, \xi \rangle \boxed{(L_\eta)_* d\mu_g} \overset{\text{def}}{=} \int \langle x, \eta^{-1}\xi \rangle\, d\mu_g(\xi)$

可以换好

$$= \langle x, \eta^{-1} \rangle\, g(x) \overset{\text{def}}{=} \int \langle x, \xi \rangle \boxed{d\mu_{\eta^{-1}g}(\xi)}$$

now by uniqueness (4.18), we have that $(L_\eta)_* d\mu_g = d\mu_{\eta^{-1}g}$

Therefore $I(L_\eta \psi) = \int \frac{\psi(\eta^{-1}\xi)}{\hat{g}(\xi)}\, d\mu_g(\xi)$

$$= \int \frac{\psi(\eta^{-1}\xi)}{\widehat{\eta^{-1}g}(\eta^{-1}\xi)}$$

, As $\widehat{\eta^{-1}g}(\eta^{-1}\xi) \overset{\text{def}}{=} \int \overline{\langle x, \eta^{-1}\xi \rangle}\, \eta^{-1}(x)\, g(x)\, dx$

$$= \hat{g}(\xi)$$

$$\text{then} = \int \frac{\psi(\xi)}{\widehat{\eta^{-1}g}(\xi)} (L_\eta)_* d\mu_g(\xi) = \int \frac{\psi(\xi)}{\widehat{\eta^{-1}g}(\xi)}\, d\mu_{\eta^{-1}g}(\xi)$$

$$= I(\psi).$$

Therefore $I(\psi)$ induces a left-invariant measure on $\hat{G}$, thus must be a Haar measure, by uniqueness

$$I(\psi) = \int \psi(\xi)\, d\xi \Rightarrow I(\psi \hat{f}) = \int \psi \hat{f} \qquad , \forall \psi \in C_c(\hat{G})$$

$$\| \int \frac{\psi \hat{f}}{\hat{f}}\, d\mu_f = \int \psi\, d\mu_f$$

$$\Rightarrow d\mu_f = \hat{f}(\xi)\, d\xi$$

$$\Rightarrow f(x) = \int \langle x, \xi \rangle\, d\mu_f(\xi) = \int \langle x, \xi \rangle\, \hat{f}(\xi)\, d\xi$$

$\square$