还有三节课. Final 方案 → take home 2~3h 再问问. Midterm 10. Final 10 → 还没定 min 86门 ↑ 80分+

Remark: Last lecture we've shown that $\mu \in M(G)$ is invertible iff $\hat{\mu}$ has a zero

and if μ is not invertible $\Rightarrow \mu \in J$, maximal ideal (closed)

then $M(G)/J$ is a Banach Algebra

every element in $M(G)/J$ is invertible, due to maximality of J

$\Rightarrow M(G)/J \cong \mathbb{C}$ (if $\exists\, x \neq \lambda e$, then $(\lambda e - x)^{-1}$ is entire).

prop 4.18: If $\mu \in M(\hat{G})$, $\phi_\mu(x) = \int \langle x, \xi \rangle \, d\mu(\xi)$, then $\phi_\mu = 0 \Rightarrow \mu = 0$.

Recall def: $B(G) = \{\phi_\mu : \mu \in M(\hat{G})\}$ = linear span of $P(G)$

and $B^1(G) = B(G) \cap L^1(G)$, in particular, if $f \in L^1$, then $f^* * f \in B^1(G)$

很多定理都是在 $B^1(G)$ 上证

and the first version of Fourier Inversion.

Thm 4.22: (Fourier Inversion thm 1)

If $f \in B^1$, then $\hat{f} \in L^1(\hat{G})$, and $f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) \, d\xi$ — suitably normalized (差个系数), called dual measure.

$$(\Leftrightarrow d\mu_f(\xi) = \hat{f}(\xi) d\xi)$$

Example of dual measure.

① probability Haar measure on compact G ($\hat{G}$) $\leftrightarrow$ Counting measure on discrete $\hat{G}$ (G)

② $\mathbb{R}$ is self-dual by the pairing $\langle x, \xi \rangle = e^{2\pi i x \xi}$

③ $\mathbb{Q}_p$ is self-dual. moreover $\widehat{\chi_{B(1,0)}} = \chi_{B(1,0)}$ — not possible in $\mathbb{R}$ case. the reason why we prefer $\mathbb{Q}_p$ in number theory. (one of)

$B(1,0) = \mathbb{Z}_p$, integer ring. subgroup, compact

$= \sum_{j=0}^{\infty} c_j p^j$, with $\{1, \cdots, p-1\}^{\mathbb{N}}$ topology (product)

Every character $\xi_y \in \widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$, can be restricted onto $\mathbb{Z}_p$, that gives $\xi_y' \in \widehat{\mathbb{Z}_p}$

$$\widehat{\chi_{B(1,0)}}(\xi_y) = \int 1 \cdot \bar{\xi}_y \, |\chi_{B(1,0)}| = \langle 1, \xi_y' \rangle_{\mathbb{Z}_p} = \begin{cases} 1, & \xi_y|_{\mathbb{Z}_p} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$\xi_y(x) = e^{2\pi i y \cdot x} \Leftrightarrow y \in \mathbb{Z}_p$

即 $\chi_{B(1,0)}$.

(see Rudin, 1.5.2)

Now we consider an important Corollary of Thm 4.22 "characters separate points" — used later in the proof of pontrjagin duality

$\forall x, y \in G$, $\exists \xi \in \hat{G}$, $\xi(x) \neq \xi(y)$.

It sufficies to show: $\forall x_0 \in G \setminus \{e\}$, $\exists \xi \in \hat{G}$ s.t. $\xi(x_0) \neq 1$

$\exists$ symmetric neighborhood V of e s.t. $x_0 \notin V \cdot V$, then take $g(x) = \chi_V * \chi_V^* \in B^1$

By taking 4.22. $g(x_0) = \int \langle x_0, \xi \rangle |\widehat{\chi_V}(\xi)|^2 d\xi$

$\|$ 积分 >0 if $\langle x_0, \xi\rangle = 1$, $\forall \xi \in \widehat{G}$

0 contradiction. done.

Also a corollary of thm 4.22

Thm 4.26 (plancherel)

The Fourier transform on $L^1 \cap L^2$ extends uniquely to an isometry between $L^2(G)$ and $L^2(\widehat{G})$

involution. Banach $*$-algebra

proof: $\forall f \in L^1 \cap L^2$, $f^* * f \in \mathcal{B}^1$, then $|\hat{f}|^2 \in L^1$ (by thm 4.22).

and $\int |\hat{f}|^2 = f^* * f(1) \underset{\text{thm 4.22}}{=} \int |\hat{f}(\xi)|^2 d\xi$

Note that $L^1 \cap L^2$ is dense in L^2. Now we see $f \mapsto \hat{f}$ extends to $L^2(G) \to \mathcal{F}(L^2(G)) \subset L^2(\widehat{G})$

It remains to show it is onto

If $\exists \psi \in L^2(\widehat{G})$, s.t. $\int \psi \cdot \hat{f} = 0$, $\forall f \in L^1 \cap L^2$, then $\forall x$. $\int \psi \, \widehat{L_x f} = 0$

$$\| \quad \int \langle x^{-1}, \xi \rangle \underbrace{\psi(\xi) \hat{f}(\xi)}_{\in L^1(\widehat{G})} d\xi$$

$\int \langle y, \xi \rangle f(x^{-1}y) dy = \langle x, \xi \rangle \hat{f}(\xi)$?

$\Rightarrow \psi(\xi)\hat{f}(\xi) = 0$. a.e. by prev result

可换成 $\forall$ C_0 中的

$\Rightarrow \psi(\xi) = 0$, a.e. as $\mathcal{F}(L^1)$ is dense in C_0.

$\Downarrow$

Surjective. ▯

Corollary: If G is compact, then $\widehat{G}$ is an orthonormal basis for $L^2(G)$

4.3: The Pontrjagin Duality Thm

$\Phi: G \hookrightarrow \widehat{\widehat{G}}$, $\langle \Phi(x), \xi \rangle = \langle x, \xi \rangle$

Thm 4.3.2: Φ is an isomorphism between topological groups.

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结构定理

Now assuming thm 4.32

Thm 4.33 (Fourier Inversion Theorem II)

If $f \in L^1$, $\hat{f} \in L^1 \Rightarrow f(x) = \hat{\hat{f}}(x^{-1})$ a.e. (or equivalently $f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$ a.e.)

In particular, "=" hold every where if f is continuous.

measure theory: $f = 0$. a.e, f continuus $\Rightarrow f = 0$ everywhere.

proof: by def $\hat{\hat{f}}(\xi) = \int \overline{\langle x, \xi \rangle} f(x) dx = \int_{G = (\widehat{G})^\wedge} \langle x, \xi \rangle \boxed{f(x^{-1}) dx}$ see as $f(x^{-1})dx \in M(\widehat{\widehat{G}})$

by Pontrjagin duality, we could use thm 4.22

Recall thm 4.22, $f \in \mathcal{B}^1(G)$, then $f(x) = \int_{\widehat{G}} \langle x, \xi \rangle \hat{f}(\xi) d\xi$

$\|$

$L^1 \cap \{ \int_{\widehat{G}} \langle x, \xi \rangle d\mu(\xi), \ \mu \in M(\widehat{G}) \}$

Thus theorem 4.22 applies to $\hat{f} \in \mathcal{B}(\widehat{G})$, and $\hat{f}(\xi) = \int \langle x, \xi \rangle \hat{\hat{f}}(x) dx$

$\Rightarrow \int_{(\widehat{G})^\wedge} \langle x, \xi \rangle \left(f(x^{-1}) - \hat{\hat{f}}(x) \right) dx = 0$, $\forall \xi \in \widehat{G}$

Recall prop 4.18. $\mu \in M(\widehat{G})$, $\phi_\mu(x) = \int_{\widehat{G}} \langle x, \xi \rangle d\mu(\xi) = 0 \Rightarrow \mu = 0$

Now by prop 4.18, $f(x^{-1}) = \hat{\hat{f}}(x)$ a.e. $\square$

uniqueness of Fourier transform

Corollary 4.34: $\hat{\mu} = \hat{\nu}$, $\mu, \nu \in M(G) \Rightarrow \mu = \nu$

Proof: By pontryagin duality. $\hat{\mu} = \hat{\nu} \Leftrightarrow \phi_\mu = \phi_\nu$ (take $\alpha \in \hat{\hat{G}}$

$\xrightarrow{\text{Prop 4.18}} \mu = \nu$

Cor 4.36: G is compact/discrete $\Leftrightarrow$ $\hat{G}$ is discrete / compact

现在是双向的
之前是单向

Cor: $L^1(G)$ has multiplicative identity $\Leftrightarrow$ G is discrete.

Now we go back to the proof of thm 4.32 (pontryagin), we need 2 lemmas

h is the "Fourier inverse" of $\phi * \psi$

Lemma 4.30: If $\phi, \psi \in C_c(\hat{G})$, then $\phi * \psi = \hat{h}$, for some $h \in B^1(G)$.

In particular $\mathcal{F}(B^1)$ is dense in $L^p(\hat{G})$, for $p < \infty$ we only use the case $p=2$, later.

Proof: let $f(x) = \int \langle x, \xi \rangle \phi(\xi)\, d\xi$, $g(x) = \int \langle x, \xi \rangle \psi(\xi)\, d\xi$.

later we will show that $h \in L^1(G)$

$h(x) \overset{\text{def}}{=} \underbrace{\int \langle x, \xi \rangle \phi * \psi(\xi)\, d\xi}_{\text{of the form } \in B(G)} \boxed{\in B^1(G)}$

$= \iint \langle x, \xi \rangle \phi(\xi \eta^{-1}) \psi(\eta)\, d\eta\, d\xi$

$= \iint \langle x, \xi\eta \rangle \phi(\xi) \psi(\eta)\, d\eta\, d\xi$

$= f(x) \cdot g(x)$

then $f, g, h \in B$.

利用老扳由$L^2 \to L^1$的结果, we shall show $f, g \in L^2$, so $h \in L^1$

$\forall k \in L^1 \cap L^2(G)$ (test function, dense in $L^2 \Longrightarrow$ extend到 L^2)

$\int f \bar{k} = \iint \langle x, \xi \rangle \phi(\xi)\, d\xi\, \bar{k}(x)\, dx$

$\overset{\text{Fubini}}{=} \int \phi \cdot \overline{\hat{k}} \overset{\text{Cauchy-Schwarz}}{\leq} \|\phi\|_{L^2} \cdot \|\hat{k}\|_{L^2} \overset{\text{plancherel}}{=} \|\phi\|_{L^2} \cdot \|k\|_{L^2}$

$\Rightarrow f \in L^2$, similarly for g. so $h \in L^1(G)$. Now $\boxed{h \in B^1(G)}$

So by thm 4.22

$h(x) = \int_{\hat{G}} \langle x, \xi \rangle \hat{h}(\xi)\, d\xi \overset{\text{2个等式}}{=} \int_{\hat{G}} \langle x, \xi \rangle \phi * \psi(\xi)\, d\xi$

by prop 4.18 $\hat{h}(\xi) = \phi * \psi(\xi)$.

Finally $\mathcal{F}(B^1)$ is dense in L^p, as $\{\phi * \psi,\ \phi, \psi \in C_c(\hat{G})\}$ is dense in L^p. $\square$

pure pt topology

no need to be abelian

Lemma 4.31 : Suppose G is locally cpt, and $H \leq G$, a subgp. If H is locally cpt in the relative topology then H is closed.

used later

in embedding in proof of Pontrjagin

surjective to Im is closed.

Next time Lemma 4.31. Pontrjagin duality. Corollary. Bohr compact

June 10th ~ take home exam. (after the Final week).

Always searchable in stack exchange