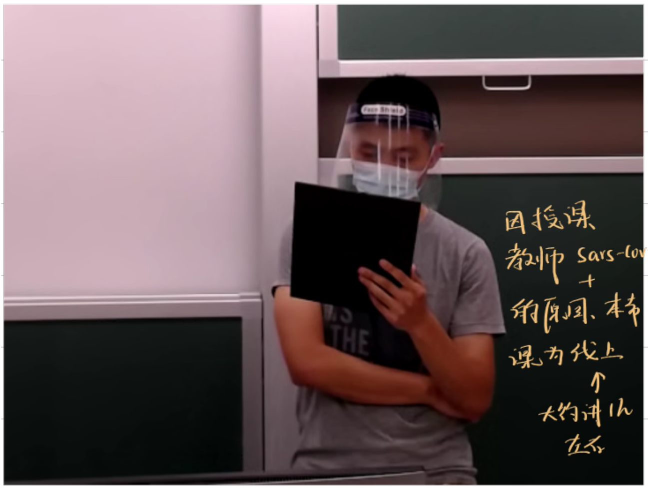


cont · Pontrjagin Duality

$\Phi: G \xrightarrow{} \widehat{\widehat{G}}$ ↑isomorphism

$x \quad \langle \Phi(x), \xi \rangle = \langle x, \xi \rangle$

and last time, we have shown

Lemma 4.30: If $\phi, \psi \in C_c(\widehat{G})$, then $\phi * \psi = \widehat{h}$

for some $h \in \mathcal{B}'(G)$.

In particular $\mathcal{F} \subset \mathcal{B}'(G)$, is dense in $L^P(\widehat{G})$

点集拓扑

Lemma 4.31 Suppose G is a locally cpt group, and $\mathcal{H}$ is a subgroup.

If $\mathcal{H}$ is locally compact in the relative topology, then $\mathcal{H}$ is closed.

proof: Notice that every relative cpt subset of $\mathcal{H}$ must be compact in G. So "$\mathcal{H}$ is locally compact in the relative topology" means that $\forall$ relative neighborhood of 1, $\exists\ \underset{\uparrow \text{open in } G}{U} \cap H$, where relative closure $K \subset H$ is a cpt subset of G

$\Downarrow$

(that implies that the closure of $U \cap H$ in G is a subset of H.)

$\exists$ symmetric neighborhood V of 1, s.t. $V \cdot V \subset U$

Say we have a $\overset{\text{sequence}}{\text{net}}$ $\underset{H}{\underset{\cap}{x_\alpha}} \to x$ in G, we need to show that $x \in H$.

since $x \in \overline{H}$, so is x^{-1}, pick $y \in Vx^{-1} \cap H$. ↗ 用y，因为x不知道x∈H？所以用y把x拖到1旁边.

↑group

Eventually x_α lies in $x \cdot V$, so $y \cdot \underset{H}{\underset{\cap}{x_\alpha}}$ lies in $Vx^{-1}xV = V \cdot V \subset U$

As $\overline{H \cap U} \subset H$, and $yx_\alpha \to yx \Rightarrow yx \in H \Rightarrow x \in H$.

$\boxed{////}$

With the above 2 lemmas, we could prove the Pontrjagin Duality

proof: As characters separate points, i.e. $\forall x \neq e$, $\exists \xi \in \widehat{G}$, s.t. $\langle x, \xi \rangle \neq 1$

$\Leftrightarrow \forall x \neq y, \exists \xi \in \widehat{G}$, s.t. $\langle x, \xi \rangle \neq \langle y, \xi \rangle$)

So Φ is $1-1$ ↗onto

Next, we show that $\Phi: G \to \Phi(G) \subset \widehat{\widehat{G}}$ is a homeomorphism, by showing

suppose $x \in G$, and $\{x_\alpha\}_{\alpha \in A}$ is a net in G, then ci) → ci$\overset{4}{\text{v}}$) are equivalent

① $x_\alpha \to x$ in G

② $f(x_\alpha) \to f(x)$, $\forall f \in \mathcal{B}'(G)$

③ $\int \langle x_\alpha, \xi \rangle \check{f}(\xi)\, d\xi \to \int \langle x, \xi \rangle \check{f}(\xi)\, d\xi$, $\forall f \in \mathcal{B}'(G)$

④ $\Phi(x_\alpha) \to \Phi(x)$ in $\widehat{\widehat{G}}$

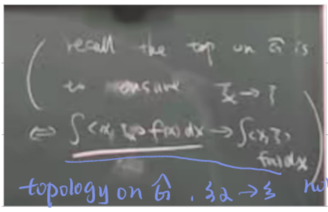
trivial.

① ⇒ ② ⟺ ③ ⟺ ④ → $\mathcal{F}(B')$ is dense in $L^1(\hat{G})$, lemma 4.30

(4)→(3) by DCT (dominated convergence thm)

by $\boxed{\text{thm } 4.22}$ $f \in B'$, then $f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$.
$\hat{f} \in L^1$

(3)→(4) needs more explanation.



(recall the top on $\hat{G}$ is to ensure $\xi_\alpha \to \xi$
$\Leftrightarrow \int \langle x, \xi_\alpha \rangle f(x) dx \to \int \langle x, \xi \rangle f(x) dx$)

topology on $\hat{G}$, $\xi_\alpha \to \xi$ holds

Since $\mathcal{F}(B')$ is dense in $L^1(G)$

(iii) $\Rightarrow \int \langle x_\alpha, \xi \rangle g(\xi) d\xi \to \int \langle x, \xi \rangle g(\xi) d\xi$, $\forall g \in L^1(\hat{G})$

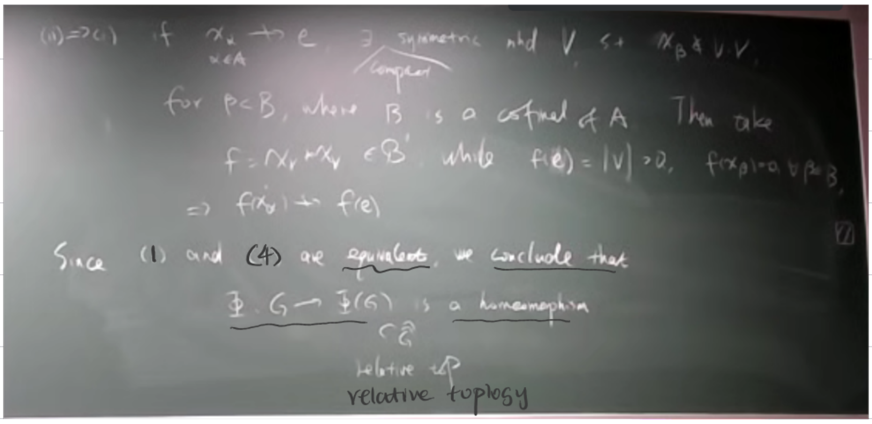
$\Rightarrow x_\alpha \to x$ on $\hat{\hat{G}}$

Now ② ⇒ ①: if $x_\alpha \not\to e$ ($\alpha \in A$), $\exists$ compact symmetric neighborhood $x_\beta \in V \cdot V$

for $\beta \in B$. where β is a cofinal? of A, then take

$f = \chi_V \cdot \chi_V \in B'$, while $f(e) = |V| > 0$, $f(x_\beta) = 0$, $\forall \beta \in B$.

$\Rightarrow f(x_\alpha) \not\to f(e)$. $\boxed{//}$



(ii)⇒(i) if $x_\alpha \not\to e$ ($\alpha \in A$), $\exists$ symmetric nhd V (compact) s.t. $x_\beta \notin V \cdot V$,
for $\beta \in B$, where B is a cofinal of A. Then take
$f = \chi_V * \chi_V \in B'$ while $f(e) = |V| > 0$, $f(x_\beta) = 0$ $\forall \beta \in B$,
$\Rightarrow f(x_\alpha) \not\to f(e)$ $\boxed{/}$

Since (1) and (4) are equivalent, we conclude that
$\Phi: G \to \Phi(G)$ is a homeomorphism
$\subset \hat{\hat{G}}$
relative top
relative toplogy

As G is locally cpt, so is $\Phi(G)$, thus closed in $\hat{\hat{G}}$ by lemma 4.31

We shall show $\Phi(G) = \hat{\hat{G}}$, If otherwise, $\exists x \in \hat{\hat{G}} \setminus \underbrace{\Phi(G)}_{\text{closed}}$

$\exists$ symmetric compact neighborhood V of e s.t. $x V \cdot V \cap \Phi(G) = \phi$

then if we take $\phi \in C_c(x \cdot V)$. $\psi \in C_c(V)$, positive, then

$$\phi * \psi |_{\Phi(G)} = 0$$
$$\| \text{ Lemma 4.30}$$
$$\hat{h} \text{ for some } h \in B'(\hat{G})$$

Therefore $0 = \phi * \psi(\Phi(x^{-1})) = \hat{h}(\Phi(x^{-1}))$

$= \int \langle \Phi(x), \xi \rangle h(\xi) d\xi$.

$= \int_{\hat{G}} \langle x, \xi \rangle h(\xi) d\xi$. , $\forall x \in G$

Prop 4.18 $\Rightarrow$ $h = 0$ a.e.

$\Rightarrow \hat{h} = 0 \Rightarrow \phi * \psi = 0$, contradiction. $\Rightarrow \Phi(G) = \hat{\hat{G}}$ $\boxed{///}$ Pontrjagin duality

Now. $\boxed{\text{Prop 4.37}}$: If $f, g \in L^2(G)$, then $(fg)^\wedge = \hat{f} * \hat{g}$

we use schwarz functions in $\mathbb{R}$ case, here we use B' functions (dense)

Proof: It suffices to assume $f, g \in L^2(G) \cap \underbrace{\mathcal{F}(B'(G))}$

dense in $L^2(G)$

then $\exists\ \phi, \psi \in B'(G)^{\cap L^2}$, s.t. $f(x) = \hat{\phi}(x^{-1})$, $g(x) = \hat{\psi}(x^{-1})$

By Fourier Inversion thm $\underline{1}$ (thm 4.22)

$$\hat{f}(\xi) = \int \overline{\langle x, \xi\rangle}\, \hat{\phi}(x^{-1})\, dx = \int \langle x, \xi\rangle \hat{\phi}(x)\, dx$$

$\Vert$ thm 4.22

$\phi(x)$

Similarly $\hat{g} = \psi$

$\Rightarrow \hat{f} * \hat{g} = \phi * \psi \quad$ = RHS

On the LHS: $fg = \hat{\phi}(x^{-1}) \cdot \hat{\psi}(x^{-1})$, by def

$= \widehat{\phi * \psi}(x^{-1}) \in L^1$

Now, by Fourier Inversion $(fg)^\wedge = \phi * \psi(x)$.

Finally, apply the density argument. $\boxed{///}$

Now, at the end, we need to explain Poisson Summation Formula

$f \in L^1(\mathbb{R})$, $\hat{f} \in L^1(\mathbb{R})$, then $\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i x \cdot n}$ In Euclidean case

in particular $\underbrace{\sum_{n\in\mathbb{Z}}}_{\text{physical}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$, 整约 x

dual lattice freq space

to General group

take $\mathbb{Z} \overset{\text{subgp}}{\leq} \mathbb{R} \Rightarrow$ Integral on subgp

Def: If H is a closed subgroup of G, define

$H^\perp = \{\xi \in \hat{G} : \underset{\forall x \in H}{\langle x, \xi\rangle = 1}\}$, a closed subgp of $\hat{G}$

Denote $q \quad G \to \underset{\text{locally cpt abelian}}{G/H}$, natural projection.

,必须H closed 才 $\underset{\sim\sim}{\text{Hausdorff}}$ (why we need closed)

Prop 4.39: $(H^\perp)^\perp = H$

Proof: $H \subset \underset{\leq G}{(H^\perp)^\perp}$ by definition. Conversely $\forall x_0 \notin H$, $q(x_0) \neq e$ in G/H

$\Downarrow$

$\exists\ \eta \in (G/H)^\wedge$ s.t. $\eta(q(x_0)) \neq 1$

$\swarrow$

It can be lifted to a character of G

$\xi(x) = \eta \circ q(x)$

$\Downarrow$

$\xi = 1$ on H, $\xi(x_0) = \eta \circ q(x_0) \neq 1$

$\Rightarrow \exists\ \xi \in H^\perp$ s.t. $\xi(x_0) \neq 1 \Rightarrow x_0 \notin (H^\perp)^\perp \Rightarrow (H^\perp)^\perp \subset H$

$\boxed{///}$

↗ prove next time

Thm 4.40: Suppose H is a closed subgp of G, then

↓

With this $\quad \Phi : (G/H)^\wedge \to H^\perp \quad , \quad \psi \ \hat{G}/H^\perp \to \hat{H}$

We could $\quad \phi(\eta) = \eta \circ q \qquad \psi(\xi H^\perp) \mapsto \xi|_H$

prove the $\quad \Phi, \Psi$ are isomorphism of topological group.

Poisson summation formula. $\quad$ ↑ 代数&拓扑 → continuity