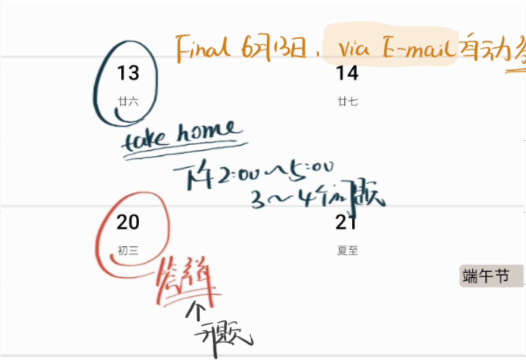


summer school ~16篇名额

Recall poisson summation formula in $\mathbb{R}$: $\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i x n}$

Def: $H$ closed subgroup of $G$, define $H^\perp \overset{\text{def}}{=} \{\xi \in \hat{G},\ \xi|_H = 1\}$

Prop 4.39: $(H^\perp)^\perp = H$

Natural projection: $q: G \to G/H$,

$$\begin{tikzcd} G \arrow[d, "q"'] \arrow[dr, dashed, "\eta\circ q"] & \\ G/H \arrow[r, "\eta"'] & S^1 \end{tikzcd}$$

提升, 投影, 限制

only this one is used later

Thm 4.40: $\Phi: (G/H)^\wedge \to H^\perp$ : $\Phi(\eta) = \eta\circ q$

$\psi: \hat{G}/H^\perp \to \hat{H}$ : $\psi(\xi\cdot H^\perp) = \xi|_H$

are isomorphisms between topological group.

Proof: Group isomorphism ✓

We first show that $\Phi$ is continuous, say $\eta_\alpha \to \eta$ in $(G/H)^\wedge$

当时定义 characters 收敛的一个等价形式

$\Leftrightarrow \eta_\alpha \to \eta$ in every compact subset $K$ of $G/H$

$\Rightarrow \eta_\alpha\circ q \to \eta\circ q$ uniformly in every compact subset of $G$

$\Rightarrow \eta_\alpha\circ q \to \eta\circ q$

Conversely, if $\eta_\alpha\circ q \to \eta\circ q$, By Lemma 2.48, $\forall$ compact $F \subset G/H$, $\exists$ a compact $K \subset G$, s.t $q(K) = F$

So $\eta_\alpha\circ q \to \eta\circ q$ uniformly on $K \Rightarrow \eta_\alpha \to \eta$ uniformly on $F$.

Hence $\Phi$ is an isomorphism.

By $\Phi$ is an isomorphism.

Now, we turn to $\psi$. By $\Phi$ $(\hat{G}/H^\perp)^\wedge \cong (H^\perp)^\perp \cong H$

$\Leftrightarrow (\hat{G}/H^\perp)^{\wedge\wedge} \cong H^\wedge$

$\|$ duality

$\hat{G}/H^\perp \cong \hat{H}$, $\Rightarrow$ Done.

More precisely. $\forall x \in H$, its corresponding element $y \in (\hat{G}/H^\perp)^\wedge$ is given by

$\langle y, \xi H^\perp\rangle = \langle x, \xi\rangle,\ \forall \xi \in \hat{G}$

$\blacksquare$

With these isomorphisms, we could calculate some dual group (used to be hard to calculate)

Example: $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ (subgp), $\xi_y(x) = e^{2\pi i x\cdot y}$, $\xi_1(x) = e^{2\pi i x}$

observe that $\mathbb{Z}_p^{\perp} = \mathbb{Z}_p = \{\sum_{j=0}^{\infty} c_j p^j, c_j = 0,1,2,\dots\}$

So by thm 4.40

$$\widehat{\mathbb{Z}_p} \cong \mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Q}_p/\ker\xi_1 \cong \text{Range}(\xi_1) = \{p^k\text{-th root of the unity}, k=1,2,\dots\}$$

$$\ker\xi_1 = \sum_{j=-1}^{\infty} c_j p^j$$

As $\mathbb{Z}_p$ is compact, $\widehat{\mathbb{Z}_p}$ is discrete

overall, $\widehat{\mathbb{Z}_p} = \{p^k\text{-th root of unity}\}_{k=1,2,\dots}$ with discrete topology

Now we could prove the Poisson summation Formula

A generalization of Fourier Inversion, see remark below

**Thm 4.43**: $f \in C_c(G)$, define $F \in C_c(G/H)$ by

$$F(xH) = \int_H f(xy)\,dy, \text{ then}$$

$$\hat{F} = \hat{f}|_{H^\perp}, \text{ where we identify } H^\perp \cong (G/H)^\wedge \ (4.40)$$

If also, $\hat{f}|_{H^\perp} \in L^1(H^\perp)$, then

$$\int_H f(xy)\,dy = \int_{H^\perp} \hat{f}(\xi)\langle x,\xi\rangle\,d\xi.$$

中间的过渡…, non-trivial.

Remark: ① If $H = \{e\}$, just Fourier Inversion

② If $G = \mathbb{R}$, $H = \mathbb{Z}$, $H^\perp = \mathbb{Z} \Rightarrow$ classical poisson summation formula on $\mathbb{R}$

use Fourier Inversion

Proof: $\forall \xi \in H^\perp = (G/H)^\wedge$, $\langle xy,\xi\rangle = \langle x,\xi\rangle$, $\forall y \in H$

$$\hat{F}(\xi) = \int_{G/H} \left(\int_H f(xy)\,dy\right) \overline{\langle xH,\xi\rangle}\,dxH$$

$\overline{\langle xH,\xi\rangle} = \overline{\langle xy,\xi\rangle}, \forall y \in H$ — Here we implicitly use $H^\perp \cong (G/H)^\wedge$

$$= \int_{G/H}\int_H f(xy)\overline{\langle xy,\xi\rangle}\,dy\,dxH$$

再 coset 分解 (thm 2.51)

$$= \int_G f(x)\overline{\langle x,\xi\rangle}\,dx = \hat{f}(\xi)$$

Finally, if $\hat{f}|_{H^\perp} \in L^1$ ($\Leftrightarrow \hat{F} \in L^1((G/H)^\wedge)$), then apply Fourier Inversion to $\hat{F}$ on $(G/H)^\wedge \cong H^\perp$

$$F(x) = \int_{(G/H)^\wedge} \hat{F}(\xi)\langle x,\xi\rangle\,d\xi = \int_{H^\perp} \hat{f}(\xi)\langle x,\xi\rangle\,d\xi$$

$$F(x) = \int_H f(xy)\,dy.$$

Now, the final section of this part

## 4.7: Bohr Compactification (non-compact $G$)

$G \to \widehat{G} \to \widehat{G_d} \to (\widehat{G_d})^\wedge \overset{\text{def}}{=} bG$

character; $\widehat{G_d}$: discrete topology (cpt 群); $(\widehat{G_d})^\wedge$: all gp homomorphisms $\widehat{G} \to S^1$ (no assumption on continuity) 由 discrete topology

$$\widehat{\widehat{G}} = \{\text{all continuous group homomorphism } \widehat{G} \to S^1\}$$
$\cong$ (SII) $G$

$G \hookrightarrow bG$ as a subgroup.

Now we show that $G$ is dense in $bG$. Consider $\overline{G} \subset bG$ (closed subgp)

$(\overline{G})^\perp = \{\text{characters on } G \text{ that is trivial on } G\} = \{1\}$
$\cap$
$\widehat{G_d} = \{\text{characters on } G \text{ with discrete topology}\}$

$\Longrightarrow$ 由 characters separate pts. $\overline{G} = [(\overline{G})^\perp]^\perp =$ the whole group $bG$.
$\Downarrow$
dense

Also, the embedding $G \hookrightarrow bG$ is continuous

$\chi_\alpha \to \chi$ in $G \Leftrightarrow \chi_\alpha(\xi) \to \chi(\xi)$, on every cpt subset of $\widehat{G}$ compact convergence topology

stronger $\Rightarrow$ embedding is continuous

and $\chi_\alpha \to \chi$ in $bG \Leftrightarrow \chi_\alpha(\xi) \to \chi(\xi)$ pointwise on $\widehat{G_d}$

discrete topology

cpt $\Rightarrow$ 有限 $\Rightarrow$ 逐点

However the embedding is not a homeomorphism.

If so, by lemma 4.31, $G$ must be closed, thus cpt $\Rightarrow$ contradiction!

closed subgp of cpt

Hard to give Example of Bohr compactification

more complicated than "single pt compactification"

**Prop 4.80**: If $K$ is a compact group, and $\rho: G \to K$, a continuous homomorphism then $\rho$ extends to a continuous homomorphism from $bG$ to $K$.

$$\begin{tikzcd} bG \arrow[dr, dashed] & \\ G \arrow[u] \arrow[r, "\rho"] & K \end{tikzcd}$$ e.g. $K = S^1$

**proof**: We may assume $K = \overline{\rho(G)}$, Abelian group, then $\rho^*: \widehat{K} \to \widehat{G}$, $\rho^*(\eta) = \eta \circ \rho$ (adjoint)

由 $K$ cpt $\Rightarrow$ discrete topology $\Rightarrow \rho^*$ is continuous from $\widehat{K}$ to $\widehat{G_d}$

take adjoint again to get a continuous group homomorphism from $bG$ to $K$. $\boxed{///}$

Bohr compactification $\Longrightarrow$ used to study almost periodic function.

**Def**: A bounded continuous function $f$ on $G$ is called **uniformly almost periodic**, if the set of translates of $f$, $\{R_x f,\ x \in G\}$, is totally bounded in the uniform metric

$\Downarrow$ Recall

($\forall \varepsilon > 0$, $\exists x_1, \dots, x_n \in G$, s.t. $\forall x \in G$, $\exists x_j$ s.t.

$\|R_x f - R_{x_j} f\|_\infty < \varepsilon$ )

$\quad = \underbrace{\|R_{x x_j^{-1}} f - f\|_\infty}_{\text{almost periodic.}}$

**Remark**: $x_j^{-1} x$ may be large, e.g. $\mathbb{R}$.

**Property**: Such an $f$ must be **uniformly continuous**: consider $K \overset{\text{def}}{=} \overline{\{R_x f,\ x \in G\}}$, compact

If $f$ is not uniformly continuous,

$\exists x_\alpha \to e \in G$, s.t. no subsequence of $R_x f$ is uniformly convergent to $f$

Since $K$ is compact, $\exists$ a subsequence $\qquad$ but $R_{x_\alpha} f \to f$ pointwise (由 $\|\cdot\|_\infty$)

$R_{x_\alpha} f \to g$ uniformly, contradiction, as it applies to $g = f$.

**Thm 4.81**: If $f$ is bounded continuous function on $G$, TFAE

① $f$ is the restriction to $G$ of a **continuous** function on $bG$

more intuitive

② $f$ is the uniform limit of linear combinations of characters on $G$

相当于 exponentials sum 的极限

③ $f$ is uniformly almost periodic.

**proof**: ①$\Rightarrow$② By **Stone–Weierstrass**, linear combinations of characters of $bG$ (compact group) are uniformly dense in $C(bG)$

So the extension of $f$ can be approximated by characters of $bG$, then restrict everything to $G$.

characters $\xRightarrow{\text{ext}}$ continuous function on $bG$

②$\Rightarrow$① : By prop 4.80, the sequence on $G$ can be extended to a sequence on $bG$

Also uniform convergence on a dense subset $\Rightarrow$ uniform convergence

①$\Leftrightarrow$③ is a little tricky (due to the definition of almost periodic)

①$\Rightarrow$③ Say $f = \phi|_G$, $\phi \in C(\underbrace{bG}_{\text{cpt}})$

Since $\underset{\substack{\cap \\ bG \\ \text{cpt}}}{x} \longmapsto \underset{\substack{\cap \\ C(bG)}}{R_x f}$ is continuous.

the set $\{R_x f,\ x \in bG\}$ is cpt in $C(bG)$, that has dense subset $\{R_x f,\ x \in G\}$, thus totally bounded

③ $\Rightarrow$ ①: Take $K = \overline{\{R_x f,\ x \in G\}}$, compact (by almost periodic) totally bdd.

By Arzela-Ascoli. $Iso(K)$ is a compact group isometric bijection

Notice $R: \underset{G}{\overset{\cap}{x}} \longmapsto \underset{Iso(K)}{\overset{\cap}{R_x}}$ continuous gp homomorphism

So it can be extended to $\tilde{R}: bG \to Iso(K)$

then $f = \tilde{R}_x f(1)|_G$, where $x \in bG$, $\boxed{///}$