X Banach space, $\mathcal{H}$ Hilbert space

1.5 complete sequences

Def: We say a sequence $\{x_n\}$ is complete in X, if $\forall x \in X$, $\forall \varepsilon > 0$, $\exists c_1, \cdots, c_n$ s.t.

$$\| x - \sum_{i=1}^n c_i x_i \| < \varepsilon$$

(may depend on ε, hence might not be a basis.)

Remark: A complete sequence may not be a basis!

e.g. ℓ^2, $x_1 = e_1$, $x_2 = e_1 + e_2$, $\cdots$, $x_n = e_1 + \cdots + e_n$

An equivalent definition is $\{x_n\}$ is complete iff

"$\mu \in X^*$, $\mu(x_n) = 0, \forall n \Rightarrow \mu = 0$ (Hahn-Banach)"

We shall discuss when a complete sequence is a basis

Exercise 1, 7 (Bonus): show that $\{\frac{1}{x+n}\}_{n=1}^{\infty}$ is complete in $L^2[0,1]$

1.6 The coefficient functionals

If $\{x_1, \cdots\}$ is a basis in X, then $\forall x = \sum_{i=1}^{\infty} c_i x_i$, so $f_n: x \longmapsto c_n$ is a linear functional, and $x = \sum_{n=1}^{\infty} f_n(x) \cdot x_n$

Thm: $f_n \in X^*$, Moreover $1 \le \|x_n\| \cdot \|f_n\| \le M$ uniform in n

proof: Since $f_n(x_n) = 1 \Rightarrow \|f_n\| \geq \frac{1}{\|x_n\|}$

Conversely, define $Y = \{(c_n)_n\}$ ($\exists \sum_{n=1}^{\infty} c_n x_n \in X$) with $\|(c_n)_n\|_Y \overset{def}{=} \sup_n \|\sum_{i=1}^n c_i x_i\|_X < \infty$. Y is a Banach space. (needs to prove)

Define $T: Y \to X$: $(c_n)_n \longmapsto \sum_{n=1}^{\infty} c_n x_n$, linear 1-1 onto, are bounded as $\|\sum_{n=1}^{\infty} c_n x_n\|_X \le \sup_n \|\sum_{i=1}^n c_i x_i\|$

Now by the open mapping theorem, T is invertible

$$\|x_n\| \cdot |f_n(x)| = \|f_n(x) \cdot x_n\| \le \|\sum_{i=1}^n f_i(x) x_i\| + \|\sum_{i=1}^{n-1} f_i(x) x_i\|$$

note that $f_n(x) \cdot x_n = \sum_{i=1}^n f_i(x) x_i - \sum_{i=1}^{n-1} f_i(x) x_i$

$$\le 2 \cdot \sup_n \|\sum_{i=1}^n f_i(x) \cdot x_i\|$$

$$\le M \cdot \|\sum_{i=1}^{\infty} f_i(x) \cdot x_i\|$$ ($= \|x\|$)

$\Rightarrow \|f_n\| \le \frac{M}{\|x_n\|}$, with $M = \|T\|^{-1}$, independent in n. ∎

Corollary: Denote $S_n(x) = \sum_{i=1}^n c_i x_i$, then $1 \le \sup_n \|S_n\| < \infty$

proof: the above argument shows

$$\|S_n(x)\| = \|\sum_{i=1}^n c_i x_i\| \le \sup_n \|\sum_{i=1}^n c_i x_i\| \le \|T\|^{-1} \cdot \|x\|.$$ ∎

Theorem: A complete sequence $\{x_n\}$ of non-zero vectors is a basis iff $\exists M > 0$ s.t. $\forall n \le m$, and scalars $c_1, \cdots, c_m$, we have

$$\|\sum_{i=1}^n c_i x_i\| \le M \cdot \|\sum_{i=1}^m c_i x_i\|$$

proof: $\Rightarrow$ since $n \le m$, then $(\sum_{i=1}^{n} c_i x_i)$ 为 $\sum_{i=1}^{m} c_i x_i$ 的部分和. then by the above Corollary

$\sup_n \|\sum_{i=1}^{n} c_i x_i\| < \infty$, as $S_n(\sum_{i=1}^{m} c_i x_i) = \sum_{i=1}^{n} c_i x_i$

$\Leftarrow$ Since $\{x_n\}$ is complete, we have $\sum_{i=1}^{n} c_i x_i \to x$, as $n \to \infty$.

From $\|(c_{kn} - c_{km}) x_k\| \le \|\sum_{i=1}^{k} (c_{in} - c_{im}) x_i\| + \|\sum_{i=1}^{k-1} (c_{in} - c_{im}) x_i\|$

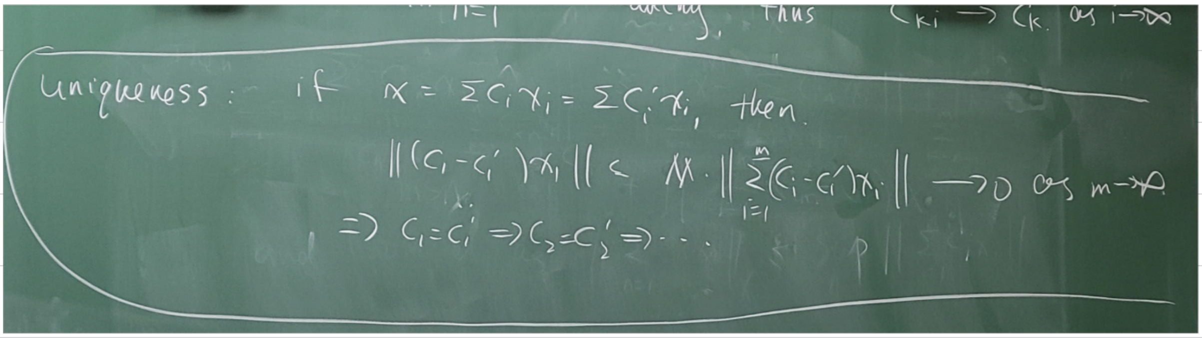
Say $m > n > k$, with k fixed. $\le M \|\sum_{i=1}^{m} (\widetilde{c_{in}} - c_{im}) x_i\|$, where $\widetilde{c_{in}} = \begin{cases} c_{in}, & i \le n \\ 0, & i > n \end{cases}$

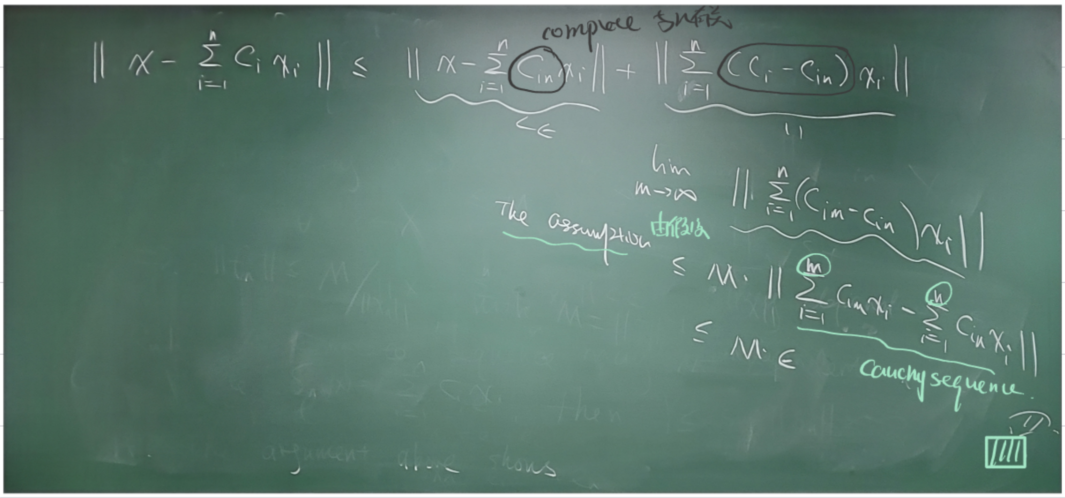
$\quad\quad \| \quad \sum_{i=1}^{n} c_{ik} x_i - \sum_{i=1}^{m} c_{im} x_i$

$\Rightarrow \forall k$ $\{c_{ki}\}_{i=1}^{\infty}$ is Cauchy thus $c_{ki} \to c_k$, as $i \to \infty$

$\uparrow$ [illegible]，不存在问题！

先证 uniqueness





接上

Exercise 1.3. (HW)

Section 1.7: Duality ($\{x_n\} \subset X$, $\{f_n\} \subset X^*$, $f_n(x_m) = \delta_{nm}$)

basis — coefficient functional.

Observation: $\{x_n\}$ is a basis $\not\Rightarrow$ $\{f_n\}$ is a basis of X^*

e.g. Exercise 1: $(\ell^1)^* = \ell^\infty$, non-separable ?

$\downarrow$

still wrong even if X^* is assumed separable

Exercise 1: $C[0,1]$ with Schauder basis.

Theorem: If $\{x_n\}$ is a basis for X, then $\{f_n\}$ is a basis for $[f_n] \stackrel{\text{def}}{=} \operatorname{span}\{f_n\}$

proof: Consider $S_n^*(f)(x) = f(S_n(x)) = f(\sum_{i=1}^{n} f_i(x) x_i) = \sum_{i=1}^{n} f_i(x) \cdot f(x_i)$

$\Rightarrow S_n^*(f) = \sum_{i=1}^n f(x_i) f_i \xrightarrow{\text{to show}} f \in [f_n]$

$\forall f \in [f_n]$, $\forall \varepsilon > 0$, $\exists g = \sum_{\text{finite}} c_i f_i$ s.t. $\|f - g\| < \varepsilon$

$\|S_n^* f - f\| \le \|S_n^*(f-g)\| + \|S_n^* g - g\| + \|f-g\|_{<\varepsilon}$

$\le M\|f-g\| < M\cdot\varepsilon$ $\quad = 0$, when n is large

uniqueness: if $0 = \sum c_i f_i$, then $0 = \sum c_i f_i(x_n) = c_n$, $\forall n$ ▯

Theorem 6: If X is reflexive, then $\{f_n\}$ is a basis for X^*

Proof: It suffices to prove $\{f_n\}$ is complete in X^*

$\forall x \in (X^*)^* = X$, if $x(f_n) = 0$, $\forall n \Rightarrow x = 0$ ▯

$\quad\quad\quad\quad\quad\quad \| f_n(x)$

Now consider Hilbert space $\mathcal{H}$, we say $\{x_n\}, \{y_n\}$ are bi-orthogonal if $(x_n, y_m) = \delta_{n,m}$ (↗总可以找正交的)

Remark: ① There exists a biorthogonal sequence of $\{x_n\}$, iff $\{x_n\}$ is minimal, i.e. $\forall n$ $x_n \notin \overline{\text{span}}_{m\neq n}\{x_m\}$

② If $\{x_n\}$ is minimal, then its biorthogonal sequence is unique iff $\{x_n\}$ is complete.

③ If $\{x_n\}$ is a basis, so is its biorthogonal basis $\{y_n\}$

(minimal + complete) (unique)

④ Let $\{f_n\}, \{g_n\}$ be bi-orthogonal basis, then $x = \sum (x, f_n) g_n = \sum (x, g_n) f_n$

$(x = \sum c_n f_n$, then $(x, g_m) = c_m)$

HW: Exercise 1.4

Main tool: Riesz basis (may not be orthogonal, but not too away from orthogonal)

↓ frequently used later in this course.

Section 1.8: Riesz Bases

Def: 2 bases are equivalent for a Banach space X, if $\exists$ a bounded invertible linear operator

$T: X \to X$ s.t. $T x_n = y_n$, $\forall n$

Thm: An equivalent def is

$$\sum_{n=1}^\infty c_n x_n \text{ is convergent} \Leftrightarrow \sum_{n=1}^\infty c_n y_n$$

Proof: "$\Rightarrow$" by definition and T

"$\Leftarrow$" let $T(\sum_{n=1}^\infty c_n x_n) = \sum_{n=1}^\infty c_n y_n$, well-defined, 1-1, onto

Consider $T_n(\sum_{i=1}^\infty c_i x_i) = \sum_{i=1}^n c_i y_i = S_n(\sum_{i=1}^\infty c_i y_i)$

$\Rightarrow \forall x$ $\sup_n |T_n(x)| \le \sup_n \|S_n\| \cdot \|y\| < \infty$

and $T_n(x) \to T(x)$

Now by the Banach–Steinhaus (共鸣定理), $\Rightarrow \|T\| < \infty$, ∎

Thm 8: In $\mathcal{H}$, equivalent bases $\{x_n\}, \{y_n\}$ have equivalent bi-orthogonal sequences $\{f_n\}, \{g_n\}$

Proof: $Tx_n = y_n$, claim $T^* g_n = f_n$

to see this $(T^* g_n, x_m) = (g_n, Tx_m) = (g_n, y_m) = \delta_{n,m}$. ∎

Def: A basis for H is called a **Riesz basis**, if it is equivalent to an orthonormal basis

$(Te_n = f_n)$
↑invertible

Remark: ① $\frac{1}{\|T^{-1}\|} \le \|f_n\| \le \|T\|$ (so $\{n \cdot e_n\}$ is not a Riesz basis)

② $\{f_n\}$ is a Riesz basis $\Rightarrow \left\{\frac{f_n}{\|f_n\|}\right\}$ is a Riesz basis

(pf: $f_n \Leftrightarrow e_n \Leftrightarrow \|f_n\| e_n$)

③ If $\{f_n\}$ is a Riesz basis, so is its biorthogonal sequence (Thm 8)

④ $\{|t|^{\alpha} e^{int}\}_{n=-\infty}^{+\infty}$, $0 < \alpha < \frac{1}{2}$, is a bounded basis but not a Riesz basis (Babenko, 1948)

Thm: In $\mathcal{H}$, TFAE (1)→(5)

(1) $\{f_n\}$ is a Riesz basis

(2) $\exists$ an equivalent inner product $(\cdot,\cdot)$ (i.e. $\|f\|_1 \approx \|f\|$)

under which $\{f_n\}$ is an orthonormal basis.

commonly used ↘

(3) $\{f_n\}$ is complete and $\exists A, B > 0$, s.t. $\forall n > 0$, $c_1, \cdots, c_n$

$A \cdot \sum_{i=1}^{n} |c_i|^2 \le \|\sum_{i=1}^{n} c_i f_i\|^2 \le B \cdot \sum_{i=1}^{n} |c_i|$

(4) $\{f_n\}$ is complete and the Gram matrix operator $(f_i, f_j)_{i,j}$ generates a

bounded invertible linear operator in ℓ^2, G in ℓ^2, where

$G(c_n)_n = (\sum_n \overline{(f_m, f_n)} c_n)_m$

(sequence, n为指标; m为指标)

(5) $\{f_n\}$ is complete, and posseses a complete bi-orthogonal sequence s.t. $\forall f \in \mathcal{H}$

$\sum |(f, f_n)|^2 < \infty$, $\sum |(f, g_n)|^2 < \infty$