cont. Equivalence of Riesz basis. 补充: $(C[0,1])^*$ is not separable e.g. $\|\delta_x-\delta_y\|=1$, $\forall x\neq y$

见 duality sec 1.7 Exercise 1 (quite difficult!)

Thm: In $\mathcal{H}$, TFAE (1)→(5)

(1) $\{f_n\}$ is a Riesz basis

(2) $\exists$ an equivalent inner product $(\cdot,\cdot)_1$ (i.e. $\|f\|_1 \approx \|f\|$)

under which $\{f_n\}$ is an orthonormal basis.

commonly used → (3) $\{f_n\}$ is complete and $\exists A,B>0$, s.t. $\forall n>0$, $c_1,\cdots,c_n$

$$A\cdot\sum_{i=1}^n |c_i|^2 \le \|\sum_{i=1}^n c_i f_i\|^2 \le B\cdot\sum_{i=1}^n |c_i|^2$$

(4) $\{f_n\}$ is complete and the Gram matrix operator $(f_i,f_j)_{i,j}$ generates a bounded invertible linear operator in ℓ^2, G in ℓ^2, where

$$\ell_2\ (c_n)_n = \left(\sum_n (f_m,f_n)\, c_n\right)_m$$

(sequence, n 为指标; m 为指标)

(5) $\{f_n\}$ is complete, and posseses a complete bi-orthogonal sequence $\{g_n\}$ s.t. $\forall f\in\mathcal{H}$

$$\sum|(f,f_n)|^2<\infty,\quad \sum|(f,g_n)|^2<\infty$$

proof: First the road map of the proof:

(5) ⇌ (1) → (4) (difficult part)

(1) → (2) → (3), (3) → (1), (4) → (3)

(1)→(2): $Tf_n=e_n$, then define an inner product $(f,g)_1 \overset{\text{def}}{=} (Tf,Tg)$ (omit: checking the inner product is equivalent)

(2)→(3): From (2): $\|\sum_{i=1}^n c_i f_i\|_1 \approx \|\sum_{i=1}^n c_i f_i\|$ (new norm from (2))

$\|\sum_{i=1}^n c_i f_i\|_1 = \sqrt{\sum_{i=1}^n |c_i|^2}$ under $(\cdot,\cdot)_1$, $\{f_n\}$ is an orthonormal basis.

(3)→(1): let $Te_n=f_n \Rightarrow \{f_n\}$ is a riesz basis

So far we have shown that (3) ← (2), (1) → (2), (3) → (1). Now for (1)→(4)

(1)→(4) completeness ✓, $Te_n=f_n$, then $(f_i,f_j)=(e_i,T^*Te_j)$, so

$$\|(\sum_j (f_i,f_j)\cdot c_j)_i\|_{\ell^2} = \|(f_i,\sum_j c_j f_j)_i\|_{\ell^2}$$

$$= \|\langle e_i, T^*T\sum_j c_j e_j\rangle_i\|_{\ell^2}$$

$$= \|T^*T\sum_j c_j e_j\|_{\mathcal{H}} \overset{\text{由T的性质}}{\approx} \|\sum_j c_j e_j\| = \sqrt{\sum_j |c_j|^2}$$

Hence we construct a bounded invertible linear operator in ℓ^2, namely G.

(4)→(3): $\|\sum_{i=1}^n c_i f_i\|^2 = (\sum_{j=1}^n c_j f_j, \sum_{i=1}^n c_i f_i) = \langle (c_i)_i, \sum_j \overline{(f_i,f_j)}\, c_j\rangle$ (Gram matrix operator G)

Recall that in Functional Analysis: If operator $G>0$ (i.e. $\langle Gf,f\rangle>0$), then $\exists\ P^2=G$, s.t. $(Gf,f)=\|Pf\|^2$

Now by the above, $\Rightarrow G\ge 0 \Rightarrow \exists\ P^2=G$, then $\|\sum c_i f_i\|^2 = \|P(c_i)\|^2 \approx \|(c_i)\|^2_{\ell^2}$

Now we show that (1), (5) are equivalent.

(1) → (5): $Te_n = f_n$, $\{f_n\}$ is a Riesz basis ⇒ so is $\{g_n\}$ $T^* g_n = e_n$

$\Rightarrow \forall f,\ f = \sum (f, f_n) g_n = \sum (f, g_n) f_n$

$\quad = (T^*)^{-1} \sum (f, f_n) e_n, \qquad = T \sum (f, g_n) e_n$

Since T is invertible (bounded) $\Rightarrow \|f\|^2 \asymp \sum |(f, f_n)|^2 \asymp \sum |(f, g_n)|^2$.

(5) → (1) (a little harder than (1) → (5))

"$\sum |(f, f_n)|^2 < \infty$" $\Rightarrow \exists C > 0$ s.t. $\sum |(f, f_n)|^2 \le C^2 \|f\|^2$

$\Sigma^1, \Sigma^2, \cdots$，取部分和构建算子族，利用 uniform boundedness thm

Similarly $\sum |(f, g_n)|^2 \le D^2 \|f\|^2$

Define $\begin{cases} S f_n = e_n, & \text{densely defined in } \{\sum_{\text{finite}} c_i f_i\} \\ T g_n = e_n & \{\sum_{\text{finite}} c_i g_i\} \end{cases}$

⇒ S, T can be extended to bounded operators with $\|S\| \le C$, $\|T\| \le D$

Recall that $S f_n = e_n \Rightarrow S^* e_n = g_n$

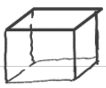
$T g_n = e_n \Rightarrow T^* e_n = f_n$

$T^* S = S T^* = Id.$ ▨

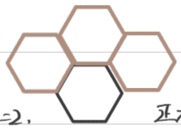
Exercise 3.5 (use Equivalent conditions 1~5)

Exponential basis for $\Omega \subseteq \mathbb{R}^n$ (domain) (Application of Riesz basis)

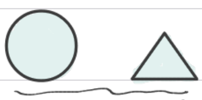
Example



Ω, $\{e^{2\pi i n \cdot x}\}_{n \in \mathbb{Z}^n}$ is an orthonormal basis



$n=2$, 正六边形. Fuglede conjecture (1974)



No basis!

"Ω has orthonormal basis iff Ω tiles $\mathbb{R}$ by translation"

FALSE! , $n \geq 3$ (Start by Terry Tao), If Ω is convex, then the conjecture is True (2019)

Question 2: Is there exponential Riesz basis on Ω $\{e^{2\pi i \lambda_n \cdot x}\}_n$

The first example of non-existence was given last year, we may leave it to the summer.

Next we shall discuss Paley-Wiener's motivation.

Question of Paley-Wiener: If perturbate $n \in \mathbb{Z}$ to λ_n, is $\{e^{2\pi i \lambda_n \cdot x}\}$ still a basis of $L^2[0,1]$

Roughly speaking, if $\{x_n\}$ is a basis, and $\{y_n\}$ is close to $\{x_n\}$, then $\{y_n\}$ is a basis equivalent to $\{x_n\}$.

Key fact: $\|I-T\|<1 \Rightarrow T$ is bounded invertible

Section 1.8: The stability of basis in Banach (with basis $\{x_n\}$)

Thm 10: If $\exists\, 0<\lambda<1$, s.t. $\forall n$, $\forall c_1, \cdots, c_n$,

$$\|\sum_{i=1}^n c_i(x_i-y_i)\| \le \lambda \|\sum_{i=1}^n c_i x_i\|,$$

then $\{y_n\}$ is a basis, equivalent to $\{x_n\}$.

Remark:

① $\lambda<1$ is necessary. i.e. $y_n=0$, $y_n=\frac{x_n}{n}$ (see Exercise 1, for a stronger version)

Proof: let $T(\sum c_i x_i) = \sum c_i(x_i-y_i)$, well-defined, bounded in X. $\|T\| \le \lambda<1$

then $\|I-(I-T)\|<1$

$\Rightarrow I-T$ is invertible, and $(I-T)x_n = y_n$. ▥

Corollary: Let $\{f_n\}$ be coefficient functionals for $\{x_n\}$, if $\sum \|f_n\|\cdot\|x_n-y_n\|<1$, then $\{y_n\}$ is a basis equivalent to $\{x_n\}$

proof: $\|\sum_{i=1}^n c_i(x_i-y_i)\| = \|\sum_{i=1}^n f_i(x)(x_i-y_i)\| \le \sum_{i=1}^n \|f_i(x)\|\cdot\|x_i-y_i\|$

$$\le \Big(\sum_{i=1}^n \|f_i\|\cdot\|x_i-y_i\|\Big)\cdot\|x\|$$

(<1, by condition) ($\|x\| = \|\sum_{i=1}^n c_i x_i\|$)

then by thm 10, $\{y_n\}$ is a basis equivalent to $\{x_n\}$. ▥

Cor of Cor above (Thm Krein-Milman-Rutman)

$\exists\, \varepsilon_n>0$, s.t. $\{y_n\}$ is a basis equivalent to $\{x_n\}$, whenever $\|y_n-x_n\|<\varepsilon_n$

Application: Recall in Lecture 1, we've constructed $\{e_n\}$ "$\wedge$" basis of $C[0,1]$.

If a Banach Space has a basis, then every dense subset contains a basis.

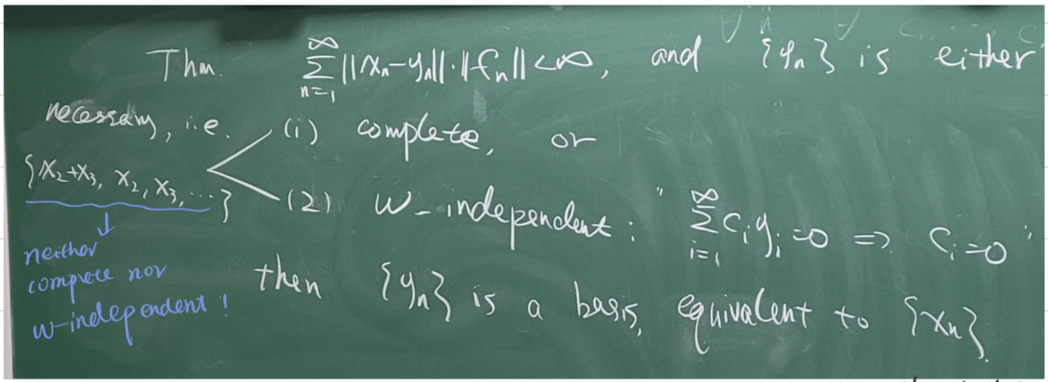
In particular, $C[0,1]$ has a polynomial basis.

Thm: $\sum_{n=1}^\infty \|x_n-y_n\|\cdot\|f_n\|<\infty$, and $\{y_n\}$ is either

(1) complete, or

(2) ω-independent: $\sum_{i=1}^\infty c_i y_i=0 \Rightarrow c_i=0$

then $\{y_n\}$ is a basis equivalent to $\{x_n\}$



Proof: By previous thm (previous corollary), one can conclude that $\exists N > 0$ s.t. ($0 + \cdots + 0 + \sum \|f_n\| \|x_n - y_n\| < 1$)

$\{x_1, \cdots, x_{N-1}, y_N, y_{N+1}, \cdots\}$ is a basis equivalent to $\{x_n\}$, then consider

$\bar{X} = X / \mathrm{span}\{y_N, y_{N+1}, \cdots\}$, space of basis $\bar{x_1}, \cdots, \bar{x_{N-1}}$ (finite dimensional space.)

$(N-1)$-dim Banach space. $\quad \|x\|_{\bar{X}} \overset{\text{def}}{=} \inf_{y - x \in \mathrm{span}\{y_N, y_{N+1}, \cdots\}} \|y\|$

assumption (1) $\Rightarrow \bar{y_1}, \cdots \bar{y_{N-1}}$ complete in $\bar{X}$

$\Rightarrow \bar{y_1}, \cdots \bar{y_{N-1}}$ is a basis for $\bar{X}$

then assumption (2) $\Rightarrow \bar{y_1}, \bar{y_2}, \cdots, \bar{y_{N-1}}$ is linearly independent

$\Rightarrow \bar{y_1}, \cdots, \bar{y_{N-1}}$ is a basis for $\bar{X}$.

Therefore $\forall y \in X$, $\exists!\ c_1, \cdots, c_{N-1}$ s.t. $y - \sum_{i=1}^{N-1} c_i y_i \in \mathrm{span}\{y_N, \cdots\}$ $\left(= \sum_{i=N}^{\infty} c_i y_i \right)$

$\Rightarrow y = \sum_{i=1}^{N} c_i y_i$, unique. ▨

Exercise 1.

Above are results in Banach space. all results in Banach space remain valid in Hilbert space.

but Hilbert space has extra structure e.g. $\|\sum c_i e_i\|^2 = \sum |c_i|^2$

Thm 13 $\|\sum c_i (e_i - f_i)\| \le \lambda \sqrt{\sum |c_i|^2}$ $(0<\lambda<1)$ $\Rightarrow \{f_i\}$ is a Riesz basis.

↑ Paley-Wiener criterion

Thm 14 (Kadec's $\frac{1}{4}$-theorem) If $\lambda_n \in \mathbb{R}$, and $|\lambda_n - n| \le L < \frac{1}{4}$, then $\{e^{i\lambda_n t}\}_n$ is a Riesz basis for $L^2[-\pi, \pi]$, moreover $\frac{1}{4}$ is sharp. with example $\lambda_n = \begin{cases} n - \frac{1}{4}, & n > 0 \\ 0, & n = 0 \\ n + \frac{1}{4}, & n < 0 \end{cases}$

Sketch of proof: $\forall \sum_j |c_j|^2 < 1$. denote $\delta_n = \lambda_n - n$, then

$\|\sum c_i e^{int} (1 - e^{i\delta_n t})\| \le 1 - \cos(\pi L) + \sin(\pi L) < 1$

and the expansion of $1-e^{iS_nt}$ relies on the orthonormal basis (正交系)

$\{1, \cos(nt), \sin(n-\frac{1}{2})t, \cdots\}$ for $L^2[-\pi,\pi]$ ~ exercises in book

Remark: by considering a different orthonormal basis for $L^2[-\pi,\pi]$, Duffin and Eachus proved that $\{e^{i\lambda_n t}\}$ is a Riesz basis if $\lambda_n \in \mathbb{C}$, $|\lambda_n - n| \le L < \frac{\log 2}{\pi}$

Another merit of Hilbert space (Cauchy-Schwarz)

Notice that $\|\sum c_i(e_i - f_i)\| \le \sum \|c_i\| \cdot \|e_i - f_i\| \overset{\text{Cauchy}}{\le} \underbrace{(\sum \|c_i\|^2)^{\frac{1}{2}}}_{= \|\sum_i c_i e_i\|} \cdot (\sum \|e_i - f_i\|^2)^{\frac{1}{2}}$

So $\sum \|e_i - f_i\|^2 < 1 \Rightarrow \{f_i\}$ is a Riesz basis.

Similar to thm 13, $\sum \|e_i - f_i\| < \infty$ + $\{f_i\}$ is either complete or ω-independent $\Rightarrow \{f_i\}$ is a Riesz basis (Bari basis)

Application: $\{\sqrt{2}\cos(n\pi t) + \frac{S_n(t)}{n+1}\}$

With $|S_n(t)|$ bounded is a Riesz basis for $L^2[0,1]$.

The Paley-Wiener criterion, namely $\|\sum c_i(e_i - f_i)\| \le \lambda\sqrt{\sum |c_i|^2}$

states that the operator $T: e_n \to f_n$ is an isomorphism, $\|I - T\| < 1$. In fact, every Riesz basis can be obtained in this way.

Thm $\{f_n\}$ is a Riesz basis (去找 $\{e_n\}$) for $\mathcal{H}$, then $\exists$ an orthonormal basis $\{e_n\}$, an isomorphism T, and $\rho > 0$, s.t.

$Te_n = \underbrace{\rho \cdot f_n}_{= g_n}$, and $\|I - T\| < 1$

Proof (Heavily relies on Functional analysis): Since $\{f_n\}$ is a Riesz, $\exists$ an orthonormal basis $\{\phi_n\}$, an isomorphism $S: \phi_n \mapsto f_n$ and $A, B > 0$

s.t. $A\sum |c_n|^2 \le \|S(\sum c_n\phi_n)\|^2 \le B\sum |c_n|^2$

let $\rho = \frac{2}{\sqrt{A}+\sqrt{B}}$, $g_n = \rho f_n$. then

$(1-\lambda)\sqrt{\sum |c_n|^2} \le \|\sum c_n g_n\| \le (1+\lambda)\sqrt{\sum |c_n|^2}$, and $\lambda = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} < 1$

then it suffices to show that $\exists$ an orthonormal basis $\{e_n\}$ s.t.

$\|\sum c_i(e_i - g_i)\| \le \lambda\sqrt{\sum_i |c_i|^2}$

$S = UP$ (polar decomposition in Functional analysis), U unitary, P positive

and $e_n = U\phi_n$, since P is self adjoint, so $I - P$ is also self-adjoint, and

$$\|I-P\| = \sup_{\|f\|=1} |((I-P)f, f)| = \sup_{\|f\|=1} \big|\, \|f\| - \underbrace{(Pf, f)}_{>0} \big|$$

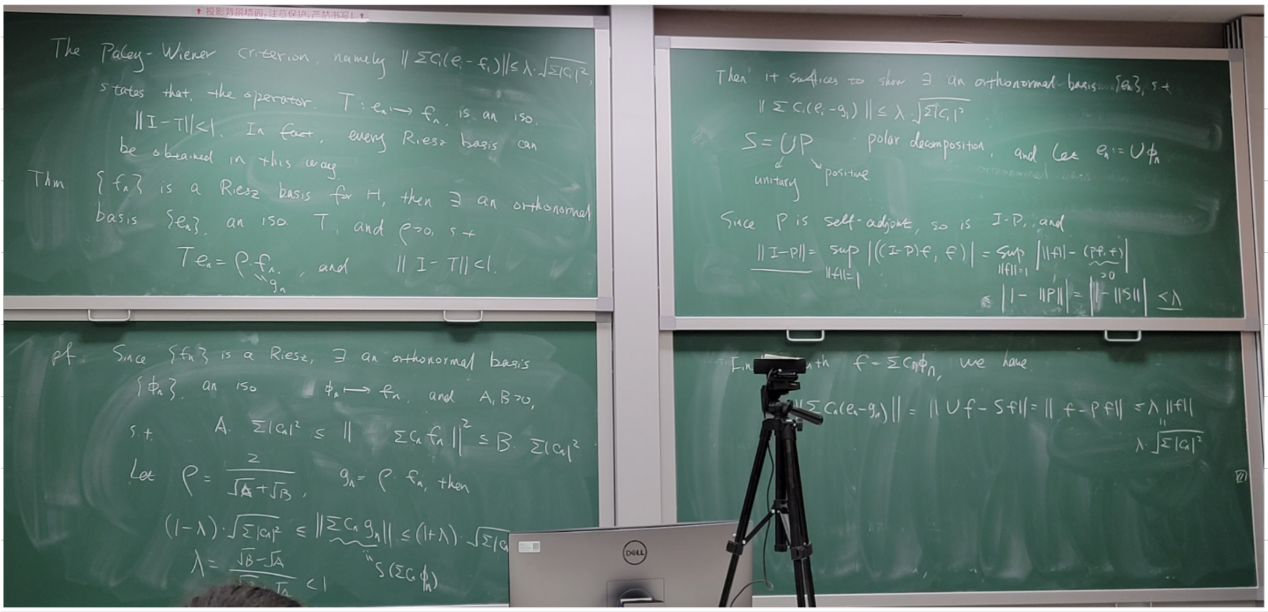
$$\leq \big|1 - \|P\|\big| = \big|1 - \|S\|\big| < \lambda .$$

↑ 可对的

Finally, with $f = \sum c_n \phi_n$, we have $\|\sum c_n (e_n - g_n)\| = \|Uf - Sf\|$ S: 对为 $\phi_n \to g_n$!

$$= \|f - Pf\|$$

$$\leq \lambda \|f\| = \lambda \sqrt{\sum |c_i|^2} \qquad \square$$



End of Chapter 1.

{ Thursday: about Assignment.

{ Next chapter mainly use Complex analysis.