03-02

1.2 Q4 $f \in C[a,b]$ Show that $\exists$ polynomials $P_1, P_2, \dots$ s.t. $f = \sum P_n$ and the series is convergent (depend on f) absolutely and uniformly

Pf. $\exists$ Q_n s.t. $\|f - Q_n\|_{L^\infty} < \frac{1}{2^n}$

Let $P_0 = Q_0$, $P_n = Q_n - Q_{n-1}$

then $f - \sum_{n=0}^{N} P_n = f - Q_N$

↑ 投影背景墙面，注意保护，严禁书写！↑

1.3 (12) An orthonormal sequence $\{e_n\}$ in $L^2[a,b]$ is complete

iff $\sum_{n=1}^{\infty} \left| \int_a^x e_n(t)\,dt \right|^2 = x-a$, $\forall x \in [a,b]$

"$\Rightarrow$" $\sum |(1_{[a,x]}, e_n)|^2 = \|1_{[a,x]}\|^2_{L^2[a,b]}$

"$\Leftarrow$" $1_{[a,x]} = \sum_{n=1}^{\infty} (1_{[a,x]}, e_n) e_n$, $\forall x$

then $1_{[x,y]} = \sum_{n=1}^{\infty} (1_{[x,y]}, e_n) e_n$

then $f = \sum_{n=1}^{\infty} (f, e_n) e_n$, $\forall$ simple function 闭子空间

which means $\text{span}(e_n)$ is dense in L^2, so $\text{span}(e_n) = L^2$

3. A function K defined on $S \times S$ is called a **positive matrix** if for each positive integer n and each choice of points $t_1, \dots, t_n$ from S the quadratic form

$$\sum_{j=1}^{n} \sum_{i=1}^{n} K(t_i, t_j) \xi_i \bar{\xi}_j$$

is positive definite.

(a) Show that the reproducing kernel of a functional Hilbert space is a positive matrix.

(b) Show that if K is a positive matrix, then there is a functional Hilbert space whose reproducing kernel is K.

1.4 ③ k on $S\times S$ is called a positive matrix, if $\forall n$, $\forall t_1,\cdots,t_n\in S$, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{n} k(t_i,t_j)\xi_i\bar{\xi}_j \text{ is positive definite}$$

(a) Show that the reproducing kernel is a positive matrix

$$k(x,y)=\sum_{n=1}^{\infty} e_n(x)\overline{e_n(y)}$$

$$\Rightarrow \sum_{i,j=1}^{n} k(t_i,t_j)\xi_i\bar{\xi}_j = \sum_{i,j,m} e_m(t_i)\overline{e_m(t_j)}\xi_i\bar{\xi}_j = \sum_m \Big|\sum_{i=1}^{n} e_m(t_i)\xi_i\Big|^2$$

利用 reproducing kernel 的定义

$$\sum k(t_i,t_j)\xi_i\bar{\xi}_j = \Big\|\sum k_{t_i}\xi_i\Big\|^2$$

$$= \sum (k_{t_j},k_{t_i})\xi_i\bar{\xi}_j = \Big\|\sum_i \xi_i k_{t_i}\Big\|^2$$

(b) Show that if k is a positive matrix on $S\times S$, then $\exists$ a functional Hilbert space ... reproducing kernel is k

Let $H=\mathrm{span}\{k_y=k(\cdot,y),\ y\in S\}$, with

$$(k_{y_1},k_{y_2})=k(y_2,y_1)$$

Stein's book, Complex Analysis, Chapter 2, Thm 5.2 — complex analysis

$\{f_n\}$ holo, $f_n\rightrightarrows f$ in every compact subset of Ω, then f is holo. in Ω.

Pf. By thm 5.1 $\int_{\text{triangle}} f=0 \Rightarrow f$ is holo for every triangle.

利用 thm 5.1 triangle 证明

Weierstrass thm.

Used later in the next chapter.

Stein的第五章

Show that $\left\{\frac{1}{x+n}\right\}_{n=1}^{\infty}$ is complete in $L^2[0,1]$.

Pf. It suffices $t^m\in\mathrm{span}\left\{\frac{1}{x+n}\right\}$, $\forall m=0,1,2,\cdots$

First $\frac{n}{x+n}\to 1$

Then, by $\boxed{\text{induction}}$, one can see $\frac{x^m}{x+n}\in$ span, x^m

利用 induction

$m=0$ ✓.

$$\frac{x^{m+1}}{x+n}=\frac{x^m(x+n)-nx^m}{x+n}=x^m-\frac{n\cdot x^m}{x+n}\in \text{span by induction hypothesis}$$

$$x^{m+1}=\lim_{n\to\infty}\frac{n\cdot x^{m+1}}{x+n}$$

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End of the QA of exercises

Chapter 2: Entire Functions of Exponential Type.

we might focus on some specific results in Complex Analysis

Why Entire? say in $C[a,b]$, if $\{e^{i\lambda_n t}\}$ is not complete, then $\exists\, \mu \in C^*[a,b]\setminus\{0\}$, s.t.

closely related to fourier transform

$$\hat{\mu}(\lambda_n) \overset{\text{def}}{=} \underbrace{\int_a^b e^{-i\lambda_n t}\, d\mu(t)}_{\text{Fourier transform}} = 0$$

$\hat{\mu}(z)$ is entire

part 1: The classical Factorization Theorems

stein's book is released later than this one, maybe better than this book

Jensen's formula: f is holomorphic in B_R, continuous in the boundary, $f \neq 0$ in $\{0\} \cup \partial B_R$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{k=1}^{n} \log\left(\frac{R}{|z_k|}\right)$$

where $z_1, \cdots, z_n$ are zeros with multiplicity.

Def: An entire function is of exponential type B, if $|f(z)| \le A\cdot e^{B|z|}$, for some $A, B > 0$.

We say that it has finite order, if $|f(z)| \le A\cdot e^{B|z|^{\rho}}$, for some $A, B, \rho > 0$

the "smallest" ρ is called the order of f: denote by $\mathrm{ord}(f)$

Note that exponential type $\neq$ order 1, e.g. $e^{|z|\cdot\log|z|}$.

Thm: Denote $n(r) \overset{\text{def}}{=}$ # of zeros in B_r, then $n(r) = O(r^{\mathrm{ord}(f)+\varepsilon})$, $\forall \varepsilon > 0$

proof: By Jensen's formula.

Def: Canonical factor of order k:

$$E_0(z) = 1 - z, \quad E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}$$

Weierstrass factorization thm:

f: entire, not identically 0, then $f(z) = z^m \cdot e^{g(z)} \cdot \prod_{n=1}^{\infty} E_n(z/z_n)$,

除法

where g is entire, $z_1, \cdots$, are non-zeros with multiplicity.

Hadamard Factorization Thm:

If f has finite order, denote $k \overset{\text{def}}{=} [\mathrm{ord}(f)]$, then

$f(z) = e^{p(z)} \cdot z^m \cdot \prod_{n=1}^{\infty} E_k(z/z_n)$, where $p(z)$ is a polynomial of order $\le k$

Example: $\sin(\pi z) = \pi z \cdot \prod_{n\neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$, mainly usage of def of order.

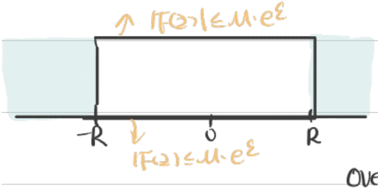
Part 2: Restriction Along a Line

Recall the Hadamard 3-Lines Lemma — quite useful, even in research!

f holomorphic and bounded in $\{0 < \operatorname{Im} z < 1\}$, continuous in the boundary and $|f(x)|, |f(x+i)| \le M$, $\forall x$, then $|f(z)| \le M$ in this strip.

边界上的估计

proof: let $F(z) = e^{-\varepsilon z^2} f(z)$, analytic, $|F(z)| = e^{-\varepsilon x^2 + \varepsilon y^2} \cdot |f(z)| \to 0$, as $|x| \to \infty$



$\exists R > 0$, s.t. $|F| \le M$, outside $\{|x| < R\}$, in $[-R, R] \times [0, 1]$, we can apply maximum principle to conclude that $|F(z)| \le M \cdot e^{\varepsilon}$

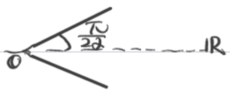
Overall, $|f(z)| \le e^{\varepsilon(x^2 - y^2) + \varepsilon} \cdot M$, $\to M$ when $\varepsilon \to 0$. ▥

the above result will be used frequently later.

Thm 6 (Phragmén–Lindelöf)

$\angle \frac{\pi}{\alpha}$ f analytic, continuous in the boundary, $|f(z)| \le M$ in the boundary, and has order $< \alpha$ inside the sector, then $|f(z)| \le M$, in this sector.

Proof: First assume



, and let $g(z) = e^{-\varepsilon z^{\gamma}} f(z)$, where $\operatorname{order}(f) < \gamma < \alpha$

gamma

$1^{\gamma} = 1$, then with $z = re^{i\theta}$.

Interior

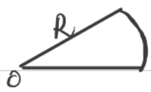
$|g(z)| = e^{-\varepsilon r^{\gamma} \cos(\gamma\theta)} |f(z)|$, since $\gamma < \alpha$, $\gamma\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\cos(\gamma\theta) \geqslant \cos(\gamma \cdot \frac{\pi}{2\alpha}) > 0$

$\le e^{-\varepsilon r^{\gamma} c} \cdot A \cdot e^{B \cdot r^{\operatorname{ord}(f) + \varepsilon}}$ exponential of f $\to 0$, when $r \to \infty$. if ε small enough s.t. $\gamma > \operatorname{ord}(f) + \varepsilon$

On boundary i.e. $\theta = \pm \frac{\pi}{2\alpha}$, $|g(z)| = e^{-\varepsilon r^{\gamma} \cos(\frac{\gamma\pi}{2\alpha})} |f(z)| \le e^{-\varepsilon r^{\gamma} \cos(\frac{\gamma\pi}{2\alpha})} \cdot M \le M$

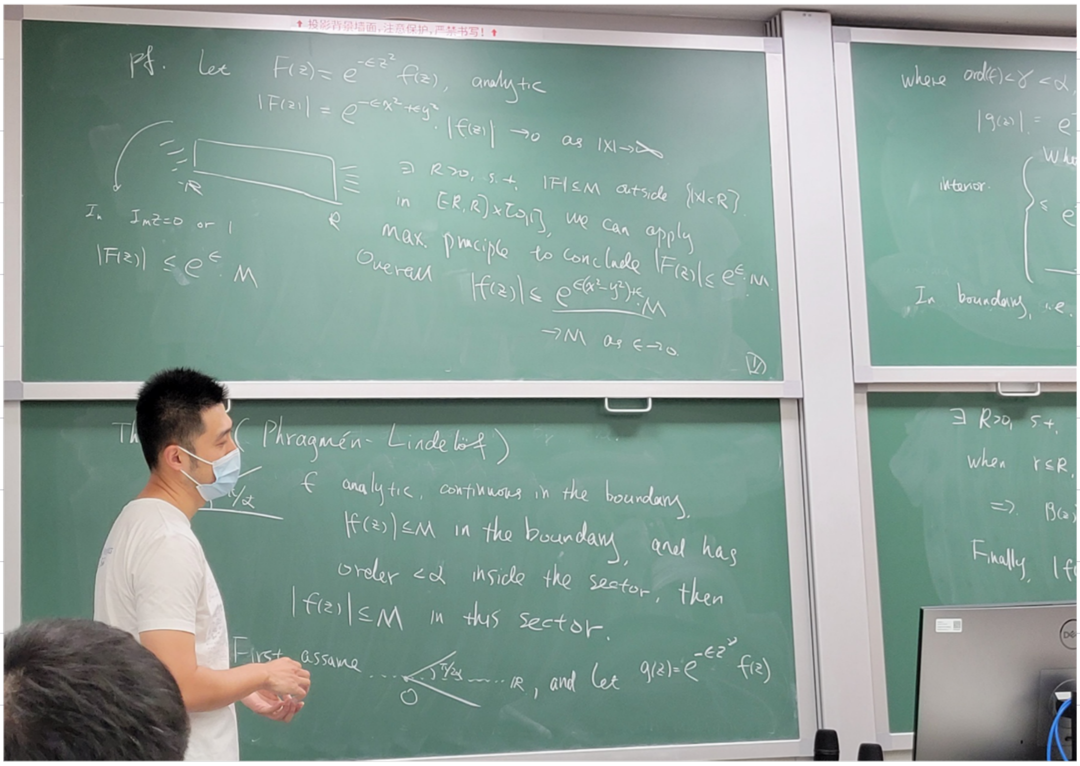
$\exists R > 0$ s.t. when $r > R$, $|g(z)| \le M$,

when $r \le R$, $|g(z)| \le M$ on the boundary of

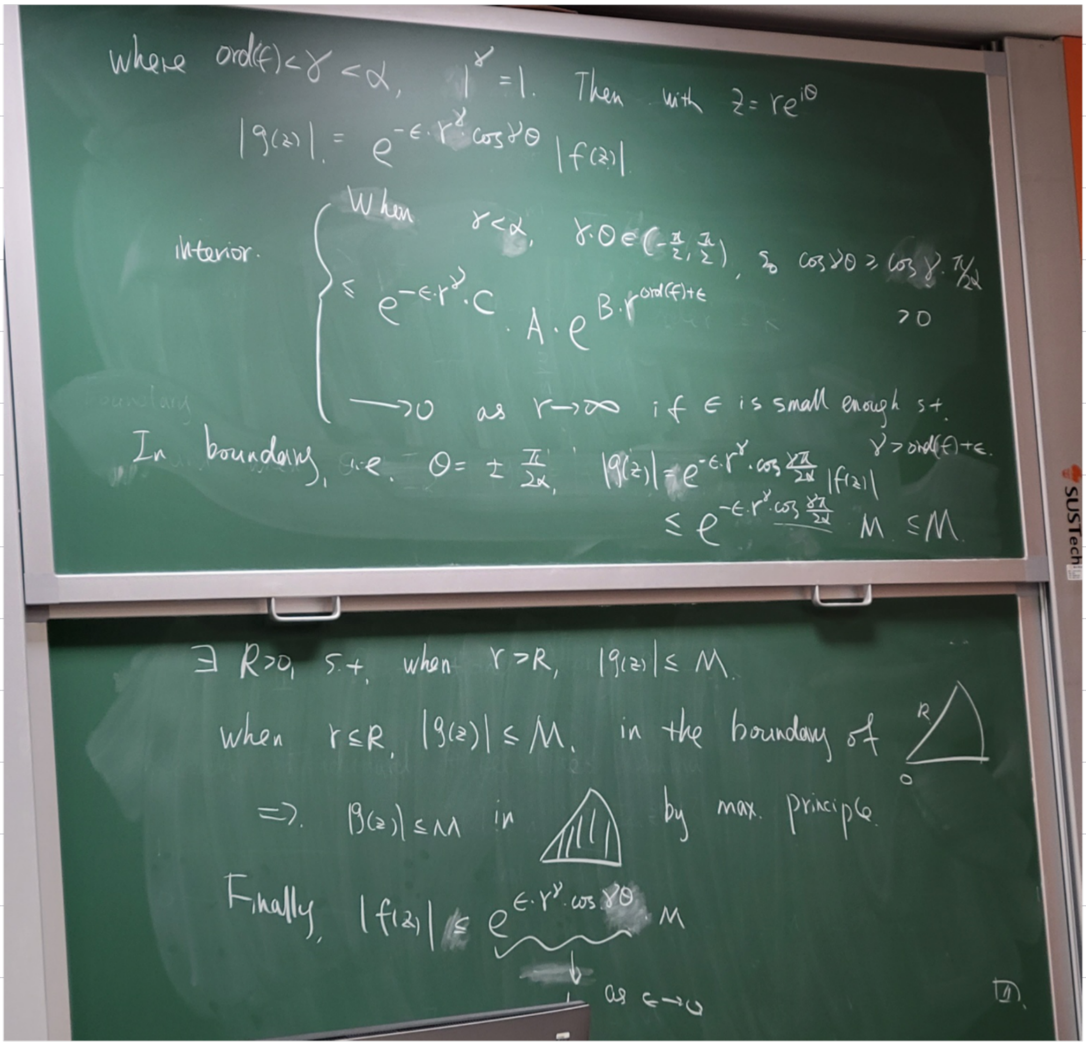


$\Rightarrow |g(z)| \le M$ in ◺ by maximum principle.

Finally, $|f(z)| \le e^{\varepsilon \cdot r^{\gamma} \cdot \cos(\gamma\theta)} \cdot M \longrightarrow 1$, as $\varepsilon \to 0$ ▥



Pf. Let $F(z) = e^{-\epsilon z^2} f(z)$, analytic
$|F(z)| = e^{-\epsilon x^2 + \epsilon y^2} \cdot |f(z)| \to 0$ as $|x| \to \infty$
$\exists R>0$, s.t. $|F| \le M$ outside $\{|x|<R\}$
in $[-R,R]\times[0,1]$, we can apply max. principle to conclude $|F(z)| \le e^{\epsilon} M$
Overall $|f(z)| \le e^{\epsilon(x^2-y^2)+\epsilon} M$
$\to M$ as $\epsilon \to 0$
$-R$
R
In $\operatorname{Im} z = 0$ or 1
$|F(z)| \le e^{\epsilon} M$
Th (Phragmén-Lindelöf)
π/α
f analytic, continuous in the boundary, $|f(z)| \le M$ in the boundary, and has order $<\alpha$ inside the sector, then $|f(z)| \le M$ in this sector.
First assume
$\pi/2\alpha$
0
$\mathbb{R}$, and let $g(z) = e^{-\epsilon z^{\gamma}} f(z)$



where $\operatorname{ord}(f) < \gamma < \alpha$, $1^{\gamma} = 1$. Then with $z = re^{i\theta}$
$|g(z)| = e^{-\epsilon \cdot r^{\gamma} \cos \gamma\theta} |f(z)|$
interior
When $\gamma < \alpha$, $\gamma\cdot\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\cos\gamma\theta \ge \cos \gamma \cdot \pi/2\alpha$
> 0
$\le e^{-\epsilon \cdot r^{\gamma} \cdot C} \cdot A \cdot e^{B \cdot r^{\operatorname{ord}(f)+\epsilon}}$
$\to 0$ as $r \to \infty$ if ϵ is small enough s.t. $\gamma > \operatorname{ord}(f)+\epsilon$.
In boundary, i.e. $\theta = \pm \frac{\pi}{2\alpha}$, $|g(z)| = e^{-\epsilon \cdot r^{\gamma} \cdot \cos \frac{\gamma\pi}{2\alpha}} |f(z)|$
$\le e^{-\epsilon \cdot r^{\gamma} \cdot \cos \frac{\gamma\pi}{2\alpha}} M \le M$
$\exists R>0$, s.t. when $r>R$, $|g(z)| \le M$.
when $r \le R$, $|g(z)| \le M$. in the boundary of
R
0
$\Rightarrow$ $|g(z)| \le M$ in
by max. principle
Finally, $|f(z)| \le e^{\epsilon \cdot r^{\gamma} \cos \gamma\theta} M$
as $\epsilon \to 0$
SUSTech