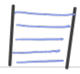
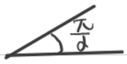
cont. Complex Analysis

① Hadamard three-line lemma 

② Phragmén-Lindelöf 

ord(f) $< \alpha$. cannot be "=" in general, e.g. e^{z^α}

maximum principle

Cor If ord(f) <1, bounded in a line, then f = constant

pf: ~ bounded in both sides of this line, then by Liouville thm. ▥

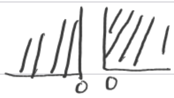
实轴上的信息、 传递到 $x+iy$

Thm 11: f entire of exponential type i.e. $|f(z)| \le A\cdot e^{B|z|}$, then $\boxed{\sup_{x\in\mathbb{R}} |f(x)| \le M} \Rightarrow |f(x+iy)| \le e^{B|y|}\cdot M$.

for $\forall y \in \mathbb{R}, x\in\mathbb{R}$

Proof: Assume $y>0$, let $g(z) \overset{\text{def}}{=} e^{i(B+\varepsilon)z} f(z)$, then $|g(x)| = |f(x)| \le M$. and $|g(iy)| = |e^{-(B+\varepsilon)y}\cdot f(iy)|$

$\le e^{-\varepsilon y} \to 0$, as $y\to +\infty$

so $N \overset{\text{def}}{=} \sup_{y\ge 0} |g(iy)|$ can be attained. Apply thm 10. to  , then we have

$\sup_{\mathrm{Im}(z)\ge 0} |g(z)| \le \max\{M, N\}$.

Notice that N cannot be larger than M. so $\sup_{\mathrm{Im}\, z\ge 0} |g(z)| \le M$

$\Rightarrow |f(x+iy)| \le |e^{iBz}|\cdot |g(z)| \le M\cdot e^{B|y|}$. ▥

Remark:

① By thm 11, $B_\tau \overset{\text{def}}{=} \{f$ entire of exponential type $\tau\}$. is a Banach space under $\|f\| \overset{\text{def}}{=} \sup_{x\in\mathbb{R}} |f(x)|$

$\boxed{\text{Bernstein's inequality}}$. $\forall f\in B_\tau$, $\|f'\| \le \tau \|f\|$, and "=" holds if and only if $f = \alpha\cdot e^{i\tau z} + \beta\cdot e^{-i\tau z}$,

$\alpha, \beta \in \mathbb{C}$

see

Exercise 12, 13, not strictly required. (Not in exam)

Thm 12: If f is entire of exponential type. and $\lim_{|x|\to\infty} |f(x)| = 0$, then $\lim_{|x|\to\infty} |f(x+iy)| = 0$, uniform in y

in every bounded set.

not covered in UG1 complex analysis

Proof: Recall $\boxed{\text{montel's thm}}$ (thm 3.3, chapter 8, stein)

not given in textbook

Suppose. $\mathcal{F} = \{f_\alpha\}_\alpha$ is a family of holomorphic functions on Ω, that is uniformly bounded in every compact subset of Ω, then

(1) $\mathcal{F}$ is equi-continuous in every cpt subset of Ω

$\forall \varepsilon>0, \exists \delta>0$, s.t. $|f_\alpha(z_1) - f_\alpha(z_2)| < \varepsilon$, $\forall \alpha$, $\forall |z_1 - z_2| < \delta$

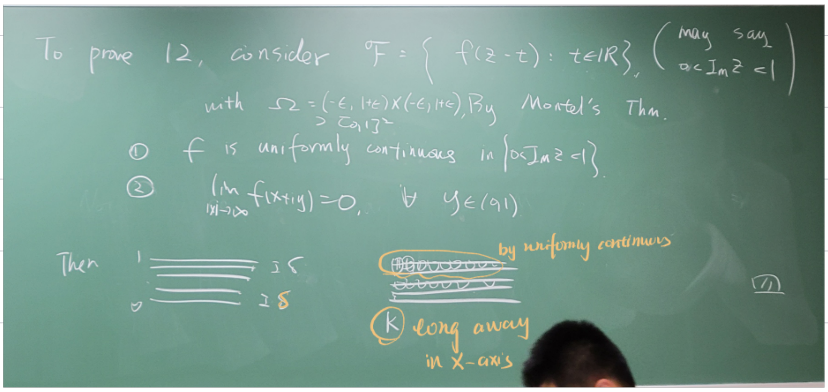
(2) $\mathcal{F}$ is a normal family (∀ sequence in $\mathcal{F}$, ∃ subsequence s.t. uniformly convergent in every compact subset)

now, to prove thm 12, consider $\mathcal{F}=\{f(z-t): t\in\mathbb{R}\}$ (may say $0<\operatorname{Im} z<1$), with

$\Omega \overset{\text{rcpt}}{=} (-\varepsilon,1+\varepsilon)\times(-\varepsilon,1+\varepsilon)\supseteq[0,1]^2$, now by montel's thm

① f is uniformly continuous in $\{0<\operatorname{Im} z<1\}$

② $\lim_{|x|\to\infty} f(x+iy)=0,\ \forall y\in(0,1)$



Exercise 2, 3, 4

meaning about proof of the 3-line lemma

Introduce a concept: Exponential type 0: $\forall \varepsilon>0, \exists A_\varepsilon>0$ s.t. $|f(z)|\le A_\varepsilon e^{\varepsilon|z|}$.

Final remark: 若为 exponential type，则可把 x-轴上信息向上平移

Section 2: Carleman's Formula

Thm 13: Let f be analytic and no zero in $\partial\Omega$ in $\{\operatorname{Im} z\ge 0\}$, $z_k=r_k e^{i\theta_k}$, $k=1,2,\cdots,n$ be its zeros in

Similar thing like Jensen's formula.

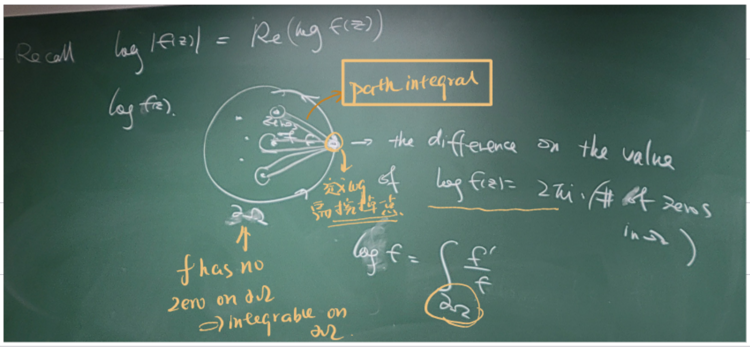
$\Omega \overset{\text{def}}{=}$  In Jensen's formula, we integrate on entire disk, hence Jensen's formula lacks information on x-axis, in thm 13, semi-disk $\Rightarrow$ Information on x-axis

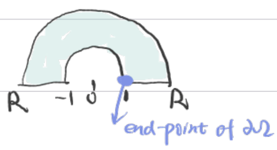
then $\sum_{k=1}^{n}\left(\frac{1}{r_k}-\frac{r_k}{R}\right)\cdot\sin\theta_k=\frac{1}{\pi R}\int_0^{\pi}\log|f(Re^{i\theta})|\sin\theta\, d\theta$

$+\frac{1}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log|f(x)f(-x)|+O_R(1)$.

comprehensive usage of complex analysis

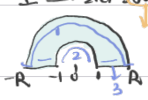
Proof: Recall that $\log|f(z)|=\operatorname{Re}(\log(f(z)))$ (property of log)





to prove thm 13, consider $I\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{1}{R^2}-\frac{1}{z^2}\right)\log(f(z))\,dz$

拆分



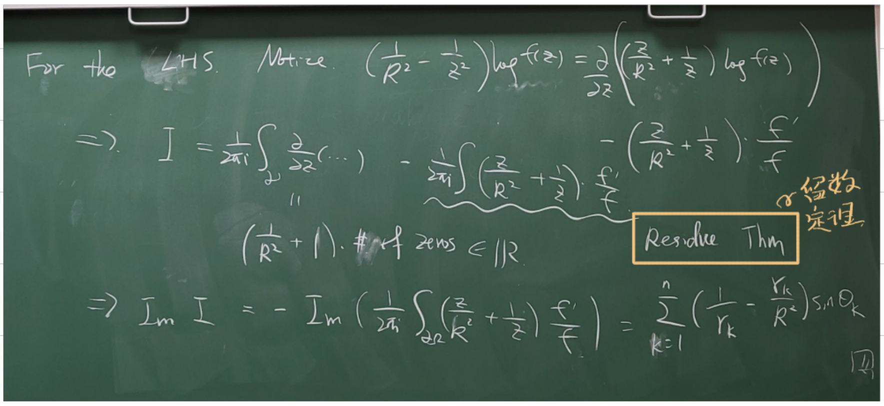
$\underbrace{\int_{\cap_R}}_{=\frac{1}{2\pi i}\int_0^{\pi}\left(\frac{1}{R^2}-\frac{1}{R^2e^{2i\theta}}\right)\log f(z)Re^{i\theta}i\,d\theta}+\underbrace{\int_{\cap_1}}_{O(1)}+\underbrace{\int_{-R}^{-1}+\int_1^R}_{=\frac{1}{2\pi i}\left[\int_{-R}^{-1}\left(\frac{1}{R^2}-\frac{1}{x^2}\right)\log f(x)dx+\int_1^R\left(\frac{1}{R^2}-\frac{1}{x^2}\right)\log f(x)dx\right)}$

So $\int_{-R}^{-1}+\int_1^R=\frac{1}{2\pi i}\int_1^R\left(\frac{1}{R^2}-\frac{1}{x^2}\right)\log f(x)f(-x)\,dx=\frac{i}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log f(x)f(-x)\,dx.$

and $\int_{\frown_{-R\,R}} = \frac{1}{2\pi i}\left[\int \frac{1}{R}\log f(z) e^{i\theta}d\theta - \int \frac{1}{R}\log f(z)\cdot e^{-i\theta}d\theta\right] = \frac{i}{\pi R}\int_0^{\pi}\log f(Re^{i\theta})\sin\theta\, d\theta.$

The RHS of Carleman can be obtained from $\mathrm{Im}(I)$ imaginary part

and for the LHS, notice that



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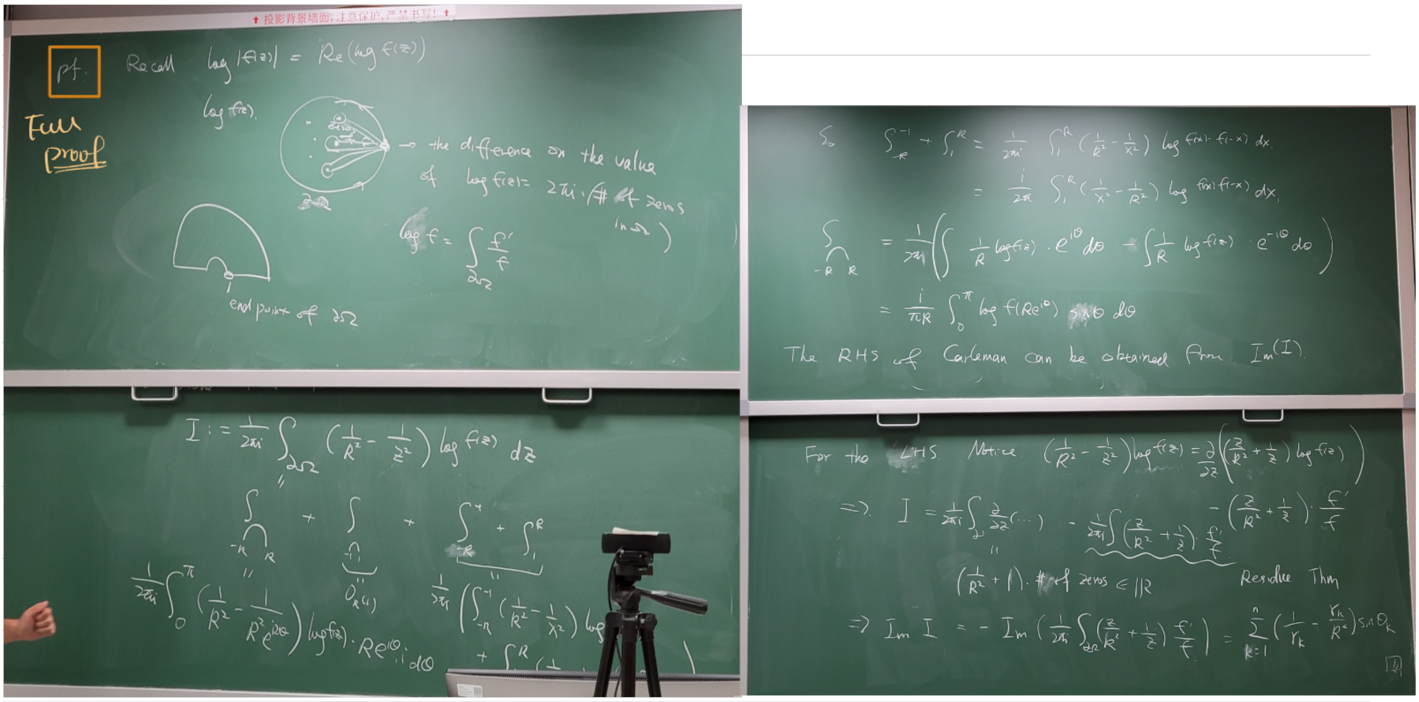


Now, we see a corollary of Carleman

Cor (Thm 14): f entire of exponential type, bounded along the real axis, then $\sum \frac{\sin\theta_k}{r_k}$ is absolutely convergent.

$z_k = r_k e^{i\theta_k}$, are zeros of f, $k=1,2,\cdots$

Proof: we may assume f has no zero in the real axis (by continuity argument) might need elaboration. then consider upper/lower half-plane separately, say $\theta_k > 0$, now, by exponential type,

$\log|f(z)| \le C\cdot|z| = C\cdot R$, so the RHS of Carleman is

$$\le \frac{1}{\pi R}\int_0^{\pi} C\cdot R\cdot\sin\theta\, d\theta + \frac{1}{2\pi}\int_1^R \left(\frac{1}{x^2}-\frac{1}{R^2}\right)\cdot C\cdot R\cdot dx + O_R(1)$$

当 $R\to\infty$ 为 $O(1)$

$$\le M < \infty, \text{ uniformly in } R$$

Which means $LHS = \sum_{k=1}^{n}\left(\frac{1}{r_k}-\frac{r_k}{R^2}\right)\sin\theta_k < M < \infty$, $\forall R$

$$\sum_{k=1}^{\infty}\left(1-\frac{r_k^2}{R^2}\right)\frac{\sin\theta_k}{r_k}\cdot\chi_{k\le n} \xrightarrow{R\to\infty} \sum_{k=1}^{\infty}\frac{\sin\theta_k}{r_k}$$

∎

we now may go back to basis

辐角

~~Cor of thm 14~~ : Thm 15 : If $\{\lambda_n\} \in \mathbb{C}$, $|\arg \lambda_n - \frac{\pi}{2}| \le L < \frac{\pi}{2}$, and $\sum \frac{1}{|\lambda_n|} = \infty$, then $\{e^{i\lambda_n t}\}$ is complete in $C[a,b]$ $\forall -\infty < a < b < +\infty$

Proof : If not, $\exists \mu \in (C[a,b])^*$ s.t. $\underbrace{f(z) = \int_a^b e^{-i\lambda_n t} d\mu(t)}_{\text{entire function of exponential type}}$ has zeros $\{\lambda_n\}$

$$\Rightarrow \sum \frac{|\sin\theta_k|}{r_k} < \infty$$

As $|\arg \lambda_n - \frac{\pi}{2}| \le L < \frac{\pi}{2}$, $|\lambda_k| \approx_L |\mathrm{Im}\lambda_k|$, or $|\sin\theta_k| \approx_L 1$.

$$\Rightarrow \sum \frac{1}{r_k} < \infty,$$ contradiction. ▥

Cor of thm 15 : $0 < \lambda_1 < \lambda_2 < \cdots$ in $\mathbb{R}$, s.t. $\sum \frac{1}{\lambda_n} = \infty$, then $\{t^{\lambda_n}\}$ is complete in $C[a,b]$, $\forall 0 < a < b < +\infty$

Proof: consider $\{i\lambda_n\}$, so by thm 15 $\{(e^{t})^{\lambda_n}\}$ is complete $\Rightarrow \{t^{\lambda_n}\}$ is complete. ▥

Section 2.2.3 Integrability on a line.

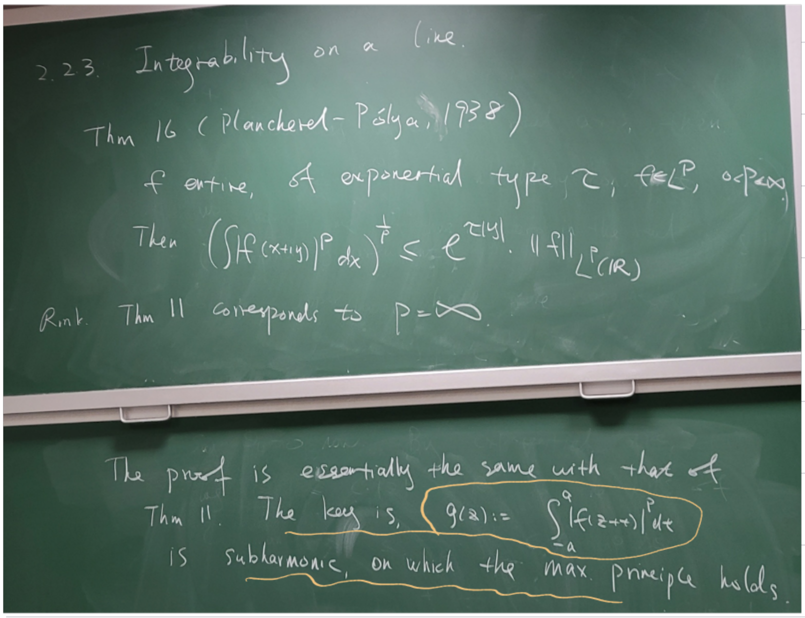
Thm 16 (Plancherel – Pólya, 1938)

later we will see L^2

f entire, of exponential type τ, $f \in L^p$, $0 < p \le \infty$, then $\left(\int |f(x+iy)|^p dx\right)^{\frac{1}{p}} \le e^{\tau|y|} \cdot \|f\|_{L^p(\mathbb{R})}$

Rmk : thm 11 corresponds to $p = \infty$

the proof is essentially the same with that of thm 11



L^2的结果要好看很多: