03-14

→ we will finish this chapter this week.

Recap: • $f(z)$ entire functions of exponential type τ, and condition on $f(x)$, $x \in \mathbb{R}$

$$|f(x)| \le M \Rightarrow |f(x+iy)| \le e^{\tau|y|} \cdot M \quad (|f(z)| \le A \cdot e^{\tau|z|})$$

- $|f(x)| \to 0$, as $|x| \to \infty$, then $|f(x+iy)| \to 0$ as $|x| \to \infty$, uniformly in y in every bounded set.
- Careleman's formula

$$\sum_{k=1}^{n} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \sin\theta_k = \frac{1}{\pi R} \int_0^{\pi} \log|f(Re^{i\theta})| \sin\theta \, d\theta$$

$$+ \frac{1}{2\pi} \int_1^{R} \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log|f(x)f(-x)| \, dx + O(1)$$

- Relativey useful Corollary: $\sum \frac{\sin\theta_k}{r_k}$ is absolutely convergent
  - Remark. It is known that $\sum \frac{1}{r_k} = \infty$.

A similar result as the first estimate

Thm 16: $\left(\int_{-\infty}^{+\infty} |f(x+iy)|^p dx\right)^{\frac{1}{p}} \le e^{\tau|y|} \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}}$

Pf: Here we maximum principle of sub-harmonic function! The Rest are almost the same.

Skip the proof.

Thm 17: f is entire of exponential type, $f \in L^p(\mathbb{R})$, for some $0 < p < \infty$, then $\forall \varepsilon > 0$, $\exists B > 0$ s.t.

for all increasing sequence $\lambda_1 < \lambda_2 < \cdots$, $|\lambda_{k+1} - \lambda_k| \ge \varepsilon > 0$, we have

$$\sum_n |f(\lambda_n)|^p \le B \int_{-\infty}^{+\infty} |f(x)|^p dx$$

Proof: Since $|f|^p$ is sub harmonic, we have

$$|f(z_0)|^p \le \frac{1}{\pi\delta^2} \iint_{|z-z_0| \le \delta} |f(z)|^p dx\, dy.$$

Take $\delta = \frac{\varepsilon}{2}$, so $\{|z-\lambda_n| < \delta\}$ are disjoint (by $|\lambda_{k+1} - \lambda_k| \ge \varepsilon > 0$), then

$$\sum |f(\lambda_n)|^p \le \frac{1}{\pi\delta^2} \cdot \iint_{|\mathrm{Im} z| \le \delta} |f(z)|^p$$

$$= \frac{1}{\pi\delta^2} \int_{-\delta}^{+\delta} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx\, dy \le \underbrace{\boxed{\frac{2 \cdot e^{\tau \cdot |\delta| \cdot p}}{\pi\delta^2}}}_{\text{constant}} \cdot \|f\|_{L^p(\mathbb{R})}^p$$ ▥

Exercise 7 is of similar principle: $\underbrace{\|f'\|_{L^p(\mathbb{R})}}_{\text{Include derivative, may use thing like Cauchy integral formula.}} \le B \cdot \|f\|_{L^p}$

2.4.4. The Paley-Wiener Thm

- $p=2$ is special, Plancherel, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$
- If $\phi(t) \in L^2[-A, A]$, then $f(z) = \int_{-A}^{A} \phi(t) e^{izt} dt$ is entire function of exponential type A, and $f \in L^2(\mathbb{R})$

Thm 18: Let $f(z)$ be an entire function s.t.

$$|f(z)| \le C \cdot e^{A \cdot |z|}, \text{ and } \int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$$

then $\exists \phi \in L^2[-A, A]$ s.t. $f(z) = \int_{-A}^{A} \phi(t) e^{izt} dt$

Stein has given a special in his complex analysis (without using real analysis)

Cor: $|f(z)| \cdot e^{-A|y|} \to 0$, as $|z| \to \infty$

proof: since $f(z) = \int_{-A}^{A} \phi(t) e^{izt} dt$, $\forall \varepsilon > 0$, $\exists \psi \in C_0^{\infty}[-A, A]$, s.t. $\|\phi - \psi\|_{L^1} < \varepsilon$

$$= \int_{-A}^{A} \psi(t) e^{izt} dt + \underbrace{\int_{-A}^{A} (\phi - \psi) e^{izt} dt}_{\leq e^{A|y|} \cdot \varepsilon}$$

$$\leq C_N \cdot \frac{e^{A|y|}}{(1+|z|)^N}$$

$\Rightarrow \lim_{|z| \to \infty} |f(z)| \cdot e^{-A|y|} \leq \varepsilon$, $\forall \varepsilon > 0$. ▨

Example: If $f \in B_\tau$, entire of exponential type τ, $\sup_{x \in \mathbb{R}} |f(x)| < \infty$,

then $f(z) = f(0) + z \int_{-\tau}^{\tau} \phi(t) e^{izt} dt$, $\phi \in L^2$

proof: $\frac{f(z) - f(0)}{z} \in L^2(\mathbb{R})$, then apply thm 18.

Example (Bernstein inequality) $\|f'\|_{L^\infty(\mathbb{R})} \leq \tau \cdot \|f\|_{L^\infty(\mathbb{R})}$

proof: consider $g_\varepsilon(z) = \underbrace{\underbrace{f(z)}_{\text{exponential type } \tau} \cdot \underbrace{\frac{\sin(\varepsilon z)}{\varepsilon z}}_{\text{exponential type } \varepsilon}}_{\text{exponential type } \varepsilon + \tau} \in L^2(\mathbb{R})$

Now by Paley-Wiener thm. $\exists \phi_\varepsilon$ s.t. $g_\varepsilon(z) = \int_{-\tau-\varepsilon}^{\tau+\varepsilon} \phi_\varepsilon(t) e^{izt} dt$

Since Bernstein can be proved for this type (Problem 12, last week), then $\|g_\varepsilon'\|_{L^\infty} \leq (\tau + \varepsilon) \|g_\varepsilon\|_{L^\infty}$,

done by $\varepsilon \to 0$. ▨

Example: $f(z)$ entire exponential type $\tau < 2\pi$, $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{+\infty} f(x) dx$$

proof: $f \in L^1 \xRightarrow{\text{see above prob}} f' \in L^1 \Rightarrow f$ is B.V. (Bounded Variation)

Poisson summation formula.

$\sum f(n) = \sum \hat{f}(n)$, where $\hat{f}(\xi) \overset{\text{def}}{=} \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \cdot \xi} dx$

But since $|f(x)| \to 0$ as $|x| \to \infty$, $f \in L^1(\mathbb{R}) \Rightarrow f \in L^2(\mathbb{R})$. by Paley-Wiener, $f(z) = \int_{-\tau}^{\tau} \phi(t) e^{izt} dt$

$\Rightarrow \hat{f} = 0$ outside $[-\frac{\tau}{2\pi}, \frac{\tau}{2\pi}] \subsetneq (-1, 1) \Rightarrow \hat{f}(n) = 0$, $n \neq 0$

$\Rightarrow \sum f(n) = \hat{f}(0) = \int_{-\infty}^{+\infty} f(x) dx$. ▨

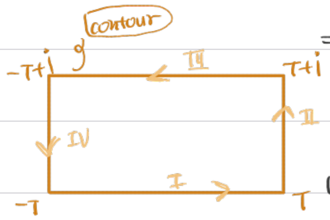
Now we go back to the proof of thm 18 (Paley-Wiener)

proof: Since $f \in L^2(\mathbb{R})$, $\phi(t) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixt} dx$ is a well-defined L^2-function. It suffices to prove

$\operatorname{supp} \phi \subseteq [-A, A]$. If so, by Fourier Inversion

$f(x)=\int_{-A}^{A}\phi(t)e^{itx}dt$ that extends to $\mathbb{C}$ $\Rightarrow f(z)=\int_{-A}^{A}\phi(t)e^{izt}dt$

Now, say $t<-A$, $\phi(t)=\int_{-\infty}^{\infty}f(x)e^{-ixt}dx$



$=\lim_{T\to\infty}\int_{-T}^{T}f(x)e^{-ixt}dx$

$0=\int_{\boxtimes}f(z)e^{izt}dz$

We need to show that $I\to0$ as $T\to\infty$

For III $\left|\int_{-T}^{T}f(x+iT)e^{-i(x+iT)t}dx\right|\le e^{Tt}\int_{-T}^{T}|f(x+iT)|dx$

cauchy-schwarz

$\le e^{Tt}\sqrt{2T}\left(\int_{-\infty}^{+\infty}|f(x+iT)|^2dx\right)^{\frac12}$

$\le e^{At}\|f\|_{L^2(\mathbb{R})}$

$\to0$, as $T\to\infty$, since $t<-A$

For II

$\int_0^T f(T+ix)e^{-i(T+ix)t}dx$, $\forall\varepsilon>0$, $\exists R$ s.t. $e^{(A+t)R}<\varepsilon$

$\int_0^T=\underbrace{\int_0^R}+\int_R^\infty$

$\le e^{Rt}\int_0^R|f(T+ix)|dx$

$\to0$, as $T\to\infty$.

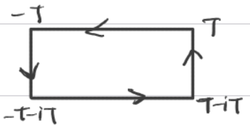
$\le\int_R^T\underbrace{|f(T+ix)|}e^{xt}dx$

$\le e^{Ax}\|f\|_{L^\infty(\mathbb{R})}$

$\le\|f\|_{L^\infty(\mathbb{R})}\cdot\underline{e^{(A+t)R}}$

$\le\varepsilon\cdot\|f\|_{L^\infty(\mathbb{R})}$

IV is the same as II, Hence $\boxed{\phi(t)=0,\ \forall t<-A}$. Now For $t>A$. consider



▨

2.2.5 The Paley-Wiener Space $PW_{[-\pi,\pi]}$

Recall $PW_{[-\pi,\pi]}\overset{\text{def}}{=}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)e^{izt}dt,\ \phi\in L^2[-\pi,\pi]\right\}$

By Paley-Wiener thm $\overset{①}{=}\{f$ entire, exponential type (π), $f\in L^2(\mathbb{R})\}$

$\langle f,g\rangle_{PW}=\langle\phi_f,\phi_g\rangle_{L^2[-\pi,\pi]}$

②

- Notice the convergence in PW implies uniform convergence.

$|f(x+iy)|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)e^{izt}dt\right|$

$\le c\,e^{|y|\cdot\pi}\cdot\underbrace{\|\phi\|_{L^2[-\pi,\pi]}}_{=\|f\|_{PW}}$

③ Since $\{e^{int}\}_{n\in\mathbb{Z}}$, is an orthonormal basis for $L^2[-\pi,\pi]$, then

$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}\cdot e^{izt}dt=\frac{\sin\pi(z-n)}{\pi(z-n)}$ is an orthonormal basis for $PW_{[-\pi,\pi]}$, therefore

$f(z)=\sum_{n=-\infty}^{\infty}C_n\cdot\frac{\sin\pi(z-n)}{\pi(z-n)}$

To compute C_n. $C_n=\langle f,\frac{\sin\pi(z-n)}{\pi(z-n)}\rangle$

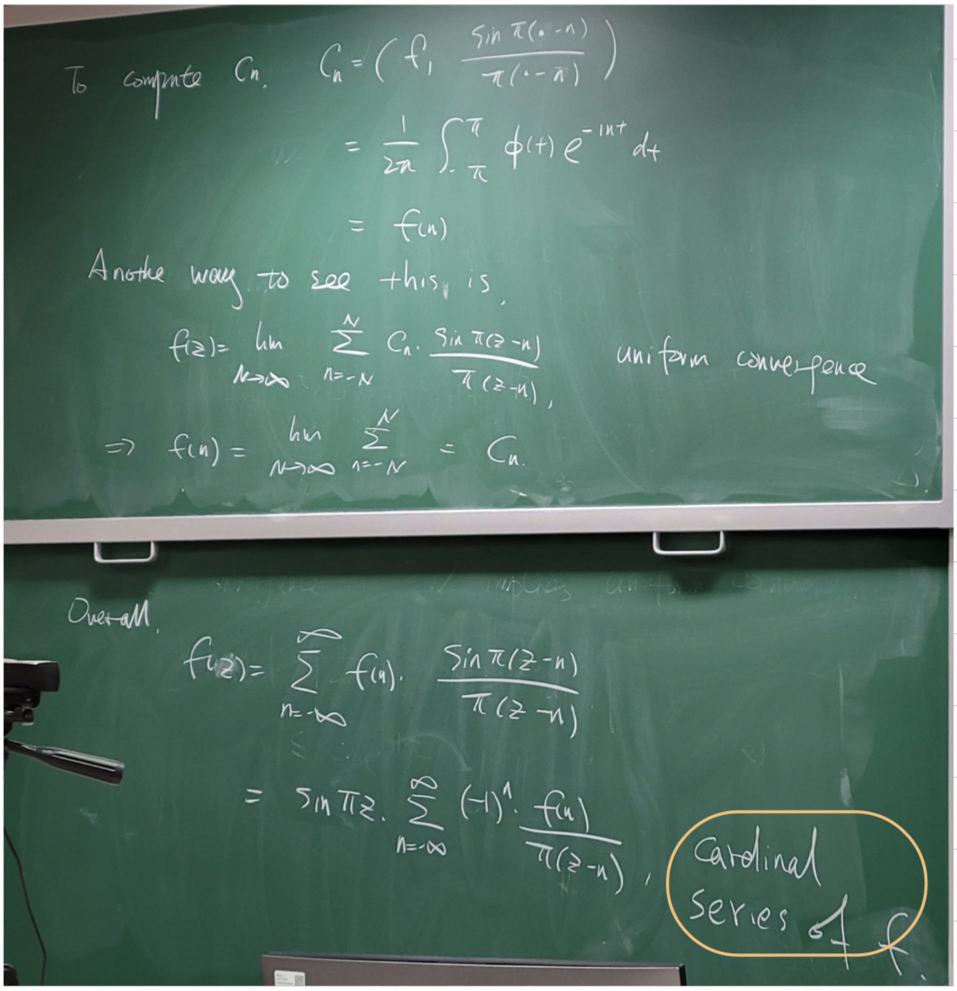
$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)e^{-int}dt=f(n)$

Another way to see this is

$f(z) = \lim_{N\to\infty} \sum_{n=-N}^{N} C_n \frac{\sin(z-n)}{\pi(z-n)}$ uniform convergence

$\Rightarrow f(n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \quad = C_n$

Overall. $f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin\pi(z-n)}{\pi(z-n)}$



To compute C_n. $C_n = \left(f, \frac{\sin\pi(\cdot - n)}{\pi(\cdot - n)} \right)$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) e^{-int} dt$

$= f(n)$

Another way to see this, is,

$f(z) = \lim_{N\to\infty} \sum_{n=-N}^{N} C_n \cdot \frac{\sin\pi(z-n)}{\pi(z-n)}$, uniform convergence

$\Rightarrow f(n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \quad = C_n.$

Overall,

$f(z) = \sum_{n=-\infty}^{\infty} f(n) \cdot \frac{\sin\pi(z-n)}{\pi(z-n)}$

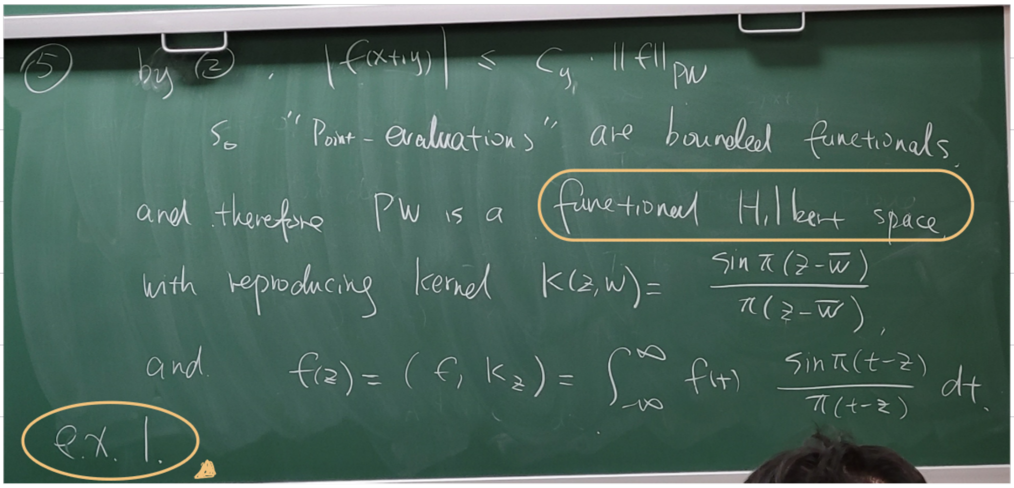
$= \sin\pi z \cdot \sum_{n=-\infty}^{\infty} (-1)^n \cdot \frac{f(n)}{\pi(z-n)}$, Cardinal series of f.

④ $f \in PW \Rightarrow f' \in PW$

$f' = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \cdot it \cdot e^{izt} dt$

$\|f'\|_{PW} = \|t \cdot \phi(t)\|_{L^2[-\pi,\pi]} \le \pi \cdot \|\phi\|_{L^2} = \pi \cdot \|f\|_{PW}$

⑤



⑤ by ②, $|f(x+iy)| \le C_y \cdot \|f\|_{PW}$

So "Point-evaluations" are bounded functionals

and therefore PW is a functional Hilbert space

with reproducing kernel $K(z,w) = \frac{\sin\pi(z-\bar{w})}{\pi(z-\bar{w})}$,

and $f(z) = (f, K_z) = \int_{-\infty}^{\infty} f(t) \frac{\sin\pi(t-z)}{\pi(t-z)} dt.$

e.x. 1.

Next week thursday. about HW for chapter 2