Chapter 3: The completeness of sets of complex exponentials

We may assume $\{\lambda_n\}$ are distinct. If some λ has multiplicity m, then every result in this chapter still holds with $\{\cdots, e^{i\lambda t}, t\cdot e^{i\lambda t}, \cdots, t^{m-1}e^{i\lambda t}, \cdots\}$

In this chapter, we may consider entire function of $\boxed{\text{Fourier transform}}$ form.

$$f(z) = \int_{-\pi}^{\pi} \phi(t)\, e^{izt}\, dt$$

Recall that $\{e^{int}\}_{n\in\mathbb{Z}}$ is complete in $L^p[-\pi,\pi]$, $1\le p<\infty$.

Def: A sequence is called **exact**, if it is complete, but fails to be complete by removing any term.

If it becomes exact when N to be removed (added), we say it has excess (deficiency) N.

周期的 ⇒ 极限也为周期的

需加入一个有区别的

Proposition: $\{e^{int}\}$ is exact in $L^p[-\pi,\pi]$, $1\le p<\infty$, but has deficiency 1 in $C[-\pi,\pi]$

proof: If we remove any e^{in_0t}, then $e^{in_0t}\in L^{p'}\setminus\{0\}$, while $\int_{-\pi}^{\pi} e^{int}\cdot e^{-in_0t}\,dt = 0$, $\forall n\ne n_0$

$\Rightarrow \{e^{int}\}_{n\in\mathbb{Z}\setminus\{n_0\}}$ is not complete. (由 complete 定义)

In $C[-\pi,\pi]$, one can see that $\{e^{int}\}_{n\in\mathbb{Z}}$ is not complete. Because in general, we do not have $f(-\pi)=f(\pi)$. To make it complete, add $e^{i\mu t}$, $\mu\in\mathbb{R}$, s.t. $e^{i\pi\mu}\ne e^{-i\pi\mu}$.

Then $\forall f\in C[-\pi,\pi]$,

consider $F(t) = f(t) - c\cdot e^{i\mu t}$, where $F(\pi) = F(-\pi)$

$$\Downarrow$$

$$c = \frac{f(\pi)-f(-\pi)}{e^{i\pi t}-e^{-i\pi t}}$$

then F can be approximated by $\{e^{int}\}_{n\in\mathbb{Z}}$

$\blacksquare$

Proposition: $\{e^{int}\}_{n\in\mathbb{Z}}$ is in-complete in every $L^p[-\pi-\varepsilon, \pi+\varepsilon]$

proof: Consider $\phi(t) = \begin{cases} -1, & -\pi-\varepsilon\le t\le -\pi+\varepsilon \\ 0, & \text{otherwise} \\ 1, & \pi-\varepsilon\le t\le \pi+\varepsilon \end{cases}$, then

构造一个

$$\int_{-\pi-\varepsilon}^{\pi+\varepsilon} \phi(t)e^{int} = -\int_{-\pi-\varepsilon}^{-\pi+\varepsilon} e^{int}dt + \int_{\pi-\varepsilon}^{\pi+\varepsilon} e^{int}dt = -\int_{\pi-\varepsilon}^{\pi+\varepsilon} e^{-int}dt + \int_{\pi-\varepsilon}^{\pi+\varepsilon} e^{int}dt = (2i)\int_{\pi-\varepsilon}^{\pi+\varepsilon}\sin(nt)dt$$

$= 0$, but $\phi\ne 0$

Hence $\{e^{int}\}_{n\in\mathbb{Z}}$ is not complete (by def)

Now, we remove $n\le 0$, to obtain $\{e^{int}\}_{n>0}$

Thm 1 (Carleman, 1922)

Suppose $\lambda_n>0$, and $\limsup_{R\to\infty} \left(\frac{1}{\log R}\sum_{\lambda_n<R}\frac{1}{\lambda_n}\right) > \frac{A}{\pi}$, then $\{e^{i\lambda_n t}\}$ is complete in $C[-A,A]$.

Corollary: $\{e^{int}\}_{n=1}^{\infty}$ is complete in $C[-A,A]$
$L^p[-A,A],\ 1\le p<\infty$

proof: $\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\to 1$

Remark: Removing finitely many terms does not change anything

proof of thm 1: If not, $\exists$ B.V. (bounded variation) function $w\ne 0$ s.t.

$$f(z)=\int_{-A}^{A}e^{izt}\,dw(t)\ \text{vanishes at } \lambda_n$$

Recall the Carleman's formula to the right half-plane ($f(iz)$) 转一下

$$\sum_{r_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\le\frac{1}{\pi R}\int_0^{\pi}\left|\log|f(Re^{i\theta})|\right|\sin\theta\,d\theta$$ 纯为zeros

$O(1)$, as $|f(re^{i\theta})|=O(e^{Ar|\sin\theta|})$

$$+\frac{1}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\underbrace{\log|f(x)f(-x)|}_{\le 2A\cdot|x|+O(1)}dx+O(1)$$

Now

$$\sum_{r_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\le 2A\cdot\underbrace{\frac{1}{2\pi}\int_1^R\left(\frac{x}{x^2}-\frac{x}{R^2}\right)dx}_{=\frac{1}{2\pi}\log R+O(1)}+O(1)$$

$$\le\frac{A}{\pi}\log(R)+O(1)$$

相除

therefore $\limsup_{R\to\infty}\left[\frac{1}{\log(R)}\sum_{r_n<R}\left(\frac{1}{\lambda_n}-\frac{\lambda_n}{R^2}\right)\right]\le\frac{A}{\pi}$

It remains to show that $\limsup_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$ $=$ (the above)

Since $\lambda_n\ge 0$, then $\limsup_{R\to\infty}\left[\frac{1}{\log(R)}\sum_{r_n<R}\left(\frac{1}{\lambda_n}-\frac{\lambda_n}{R^2}\right)\right]\le\limsup_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$

Now for $\forall\beta\in(0,1)$, $\sum_{\lambda_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\ge\sum_{\lambda_k<\beta R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)$

$$=\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}\left(1-\frac{\lambda_k^2}{R^2}\right)$$

$$\ge(1-\beta^2)\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}$$

$\Rightarrow\frac{1}{\log(R)}\sum_{\lambda_k<R}\left(\frac{1}{\lambda_k}-\frac{\lambda_k}{R^2}\right)\ge(1-\beta^2)\frac{1}{\log(R)}\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}$, take $\limsup_{R\to\infty}$

then $\limsup_{R\to\infty}(1-\beta^2)\frac{1}{\log(R)}\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}=(1-\beta^2)\limsup_{R\to\infty}\frac{1}{\log(\beta R)}\sum_{\lambda_k<\beta R}\frac{1}{\lambda_k}$

$$=(1-\beta^2)\limsup_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k},\ \forall\,0<\beta<1$$

then take $\beta\to 0$, $\limsup_{R\to\infty}\text{LHS}\ge\limsup_{R\to\infty}\frac{1}{\log(R)}\sum_{\lambda_k<R}\frac{1}{\lambda_k}$
$\frac{A}{\pi}\ \text{//}$

Contradiction to our assumption! $\boxed{///}$

Exercise (必要条件): Thm If $\lambda_n>0$, $\liminf_{\lambda\to\infty}\frac{n}{\lambda_n}>\frac{A}{\pi}$, then $\{e^{i\lambda_n t}\}$ is complete in $C[-A,A]$

We may reduce it to above argument

Remark: In fact, $\limsup_{n\to\infty}\frac{n}{\lambda_n}>\frac{A}{\pi}$ is enough.

3.2 Exponentials close to the trigonometric system.

Recall Kadec's $\frac{1}{4}$-thm: $\lambda_n \in \mathbb{R}$

$\{e^{i\lambda_n t}\}_n$ is a Riesz basis for $L^2[-\pi,\pi]$, if $|\lambda_n - n| \le L < \frac{1}{4}$

proof relies on certain expansion

Thm 3: Given $\{\lambda_n\} \subseteq \mathbb{C}$, denote $n(r) \overset{def}{=} \#\{|\lambda_n| \le r\}$, and $N(r) = \int_1^r \frac{n(t)}{t}\,dt$

then $\{e^{i\lambda_n t}\}$ is $\boxed{\text{complete}}$ in $L^p[-\pi,\pi]$, $1<p<\infty$, if

$\varlimsup_{r\to\infty} (N(r) - 2r + \frac{1}{p}\log(r)) > -\infty$

Remark: If $\{\lambda_n\}$ is complete then $\forall \lambda \in \mathbb{C}$, $\{\lambda_n - \lambda\}$ is also complete.

平移不改变完备性 $\int_{-\pi}^{\pi} \phi(t) e^{i(\lambda_n - \lambda)t}\,dt = \int_{-\pi}^{\pi} (\phi(t) e^{-i\lambda t}) e^{i\lambda_n t}\,dt$

proof of thm3: If not, $\exists \phi \in L^{p'}$ s.t. $f(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt}\,dt$, vanishes at $\{\lambda_n\}$, we may assume

$\|\phi\|_{L^{p'}} = 1$

Recall that Jensen's formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \log|f(0)| = \sum_{k=1}^{n} \log\left(\frac{r}{r_k}\right) = \boxed{\int_0^r \frac{n(t)}{t}\,dt}$$

$$= N(r) + O(1)$$

We now write $f(z) = \int_{-\pi+\varepsilon}^{\pi-\varepsilon} + \int_{\pi-\varepsilon \le |t| \le \pi}$ exponential decay (ε)

now by Hölder $\le \underbrace{\|\phi\|_{L^{p'}}}_{=1} (\int_{-\pi+\varepsilon}^{\boxed{\pi-\varepsilon}} e^{|y|\cdot t\cdot p}\,dt)^{\frac{1}{p}} + \boxed{O_\varepsilon(1)} (\int_{-\pi}^{\pi} e^{|y|\cdot t\cdot p}\,dt)^{\frac{1}{p}}$ → 少了ε的exponential 小量, extra decay

$$\le C[e^{(\pi-\varepsilon)|y|}\cdot |y|^{-\frac{1}{p}} + O_\varepsilon(1)\cdot e^{\pi|y|}\cdot|y|^{-\frac{1}{p}}]$$

$$= C\cdot e^{\pi|y|}\cdot|y|^{-\frac{1}{p}}[e^{-\varepsilon|y|} + O_\varepsilon(1)]$$

Now substitute the above estimate of f into Jensen's formula

LHS $= N(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log|e^{\pi|y|}\cdot|y|^{-\frac{1}{p}}\cdot(e^{-\varepsilon|y|} + O_\varepsilon(1))|\,d\theta$, here $y = r\sin\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \pi\cdot r\cdot|\sin\theta| - \frac{1}{p}\log(r) - \frac{1}{p}\log|\sin\theta| + \log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)|\,d\theta$$

$$= 2r - \frac{1}{p}\log(r) + O(1) + \frac{1}{2\pi}\int_0^{2\pi}\log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)|\,d\theta$$

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$\Rightarrow N(r) - 2r + \frac{1}{p}\log(r) \le \frac{1}{2\pi}\int_0^{2\pi}\log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)|\,d\theta + O(1)$

now taking $\limsup_{R\to\infty}$ Note that $\limsup$ RHS $= -\liminf \int -$ → Fatou

one can take ε small s.t. $O_\varepsilon(1) < \frac{1}{2}$

then we r is large, $e^{-\varepsilon r|\sin\theta|} < \frac{1}{2}$, $\sin\theta \ne 0$

$\Rightarrow \log|e^{-\varepsilon r|\sin\theta|} + O_\varepsilon(1)| < 0$

Hence $\limsup$ RHS $= -\liminf \int -\log|\sim|$

cont. $\leq \int \limsup_{R\to\infty} \log \left| e^{-\varepsilon r|\sin\theta|} + o_\varepsilon(1) \right| d\theta$

$\sim \log(0)\, G(1)$ close to $-\infty$. $\boxed{///}$

Direct Corollary

Thm 4: $1<p<\infty$, $\lambda_n \in \mathbb{C}$, $|\lambda_n| \leq n + \frac{1}{2p}$, then $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$, the constant $\frac{1}{2p}$ is optimal.

proof: To see the completeness, notice that

$$N(r) = \int_1^r \frac{n(t)}{t}\,dt \geq \int_1^r \frac{1+2\left[t-\frac{1}{2p}\right]}{t}\,dt$$ 取整

$$= \int_1^r \frac{2(t-\frac{1}{2p})}{t}\,dt + \int_1^r \frac{1+2\left[t-\frac{1}{2p}\right]-2(t-\frac{1}{2p})}{t}$$

$$= 2r - \frac{1}{p}\log(r) + \boxed{\frac{1}{2}\int_1^r \frac{\frac{1}{2}+\left[t-\frac{1}{2p}\right]-(t-\frac{1}{2p})}{t}\,dt} + O(1)$$

finite

$\int_1^r \frac{\frac{1}{2}+[x]-x}{x}\,dx$ may seen in exercises

Hence $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$. $\boxed{///}$

Some arguments come from research papers!