Recap: thm 1: $\lambda_n > 0$, and $\limsup_{R\to\infty} \left(\frac{1}{\log(R)} \sum_{\lambda_n < R} \frac{1}{\lambda_n}\right) > \frac{A}{\pi}$, the $\{e^{i\lambda_n t}\}$ is complete in $C[-A, A]$

Rmk (mathematical analysis): $\lambda_n > 0$, $\liminf_{n\to\infty} \frac{n}{\lambda_n} > \frac{A}{\pi}$ is sufficient.

In fact $\limsup_{n\to\infty} \frac{n}{\lambda_n} > \frac{A}{\pi}$ is enough!

Thm 3: $\lambda_n \in \mathbb{C}$, $n(r) \overset{\text{def}}{=} \#\{\lambda_n : |\lambda_n| \le r\}$, $N(r) = \int_1^r \frac{n(t)}{t} dt$, then $\{e^{i\lambda_n t}\}$ complete in $L^p[-\pi, \pi]$, $1 < p < \infty$, if

$\lim_{r\to\infty} (N(r) - 2r + \frac{1}{p}\log r) > -\infty$

Direct Corollary **thm 4**: $1 < p < \infty$, $\lambda_n \in \mathbb{C}$, $|\lambda_n| \le |n| + \boxed{\frac{1}{2p}}$, then $\{e^{i\lambda_n t}\}$ is complete in $L^p[-\pi,\pi]$

constant $\frac{1}{2p}$ is best possible

quite tricky

cont. proof of the optimality of $\frac{1}{2p}$:

take $\lambda_n = \begin{cases} n + \frac{1}{2p} + \varepsilon, & n > 0 \\ 0, & n = 0 \\ n - \frac{1}{2p} - \varepsilon, & n < 0 \end{cases}$, ε is an arbitrarily positive number.

Consider $\phi(t) = [\cos(\frac{t}{2})]^{\frac{1}{p} - 1 + 2\varepsilon} \cdot \sin\frac{1}{2}t$

$\phi \in L^q[-\pi,\pi]$, as $([\cos(\frac{t}{2})]^{-\frac{1}{q} + 2\varepsilon} \sin(\frac{1}{2}t))^q$ 可积

有何的

We shall show that $\int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t} dt = 0$, $\forall n$

$\phi(t)$ orthogonal to every element $e^{i\lambda_n t}$

First $\phi \in L^{p'} = L^q = (L^p)^*$

$n = 0$, $\lambda_0 = 0$ (by def), $\int_{-\pi}^{\pi} \phi(t) dt = 0$ ✓

$n > 0$, since $\sin(\frac{t}{2}) = \frac{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}}{2i}$, $\cos(\frac{t}{2}) = \frac{e^{i\frac{t}{2}} + e^{-i\frac{t}{2}}}{2}$, the complex form, then. let $c = \frac{1}{2p} + \varepsilon$, $\phi = (\cos\frac{t}{2})^{2c-1} \sin(\frac{t}{2})$

$$\int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t} = i \cdot 2^{-2c} \int_{-\pi}^{\pi} (1 + e^{it})^{2c-1} (1 - e^{it}) e^{int} dt$$

$$= \lim_{r\to 1^-} \int_{-\pi}^{\pi} \underbrace{(1 + re^{it})^{2c-1}}_{= \sum_{k=0}^{\infty} \binom{2c-1}{k} r^k e^{ikt}} (1 - e^{it}) e^{int} dt$$

插入r后,可以展开

$$= \lim_{r\to 1^-} \sum_{k=0}^{\infty} \binom{2c-1}{k} r^k \int_{-\pi}^{\pi} e^{i(n+k)t} (1 - e^{it}) dt$$

$$= 0, \text{ for } n = 1, 2, 3.$$

$n < 0$ is of similar principle. ∎

Existence of $\phi_n \neq 0$ make $\{e^{i\lambda_n t}\}$ not complete $\Rightarrow$ optimality of $\frac{1}{2p}$

Remark: Thm 4 $\boxed{\text{fails}}$ for $p = 1$, take $\lambda_n = \begin{cases} n + \frac{1}{2}, & n > 0 \\ 0, & n = 0 \\ n - \frac{1}{2}, & n < 0 \end{cases}$, then $\sin(\frac{t}{2})$ is orthogonal to every $e^{i\lambda_n t}$.

- Thm 4 provides simple examples of sets that are complete in $L^p[-\pi,\pi]$, but fail to be complete in $L^r[-\pi,\pi]$, $r > p$. (对有限测度空间 L^p 的包含关系)

Recall the Kadec-$\frac{1}{4}$-thm

counter-example. (complete but not a basis)

3.3: Counter example $\lambda_n = \begin{cases} n-\frac14, & n>0 \\ n+\frac14, & n<0 \end{cases}$, no term associated to $n=0$

Thm 5 $\{e^{\pm i(n-\frac14)t}\}_{n>0}$ is exact in $L^2[-\pi,\pi]$, but not a (Riesz) basis

proof: First show it is complete, to do this we translate $\underbrace{\{\pm(n-\frac14)\}}_{\Vert}$ by $\frac12$ to

$\{\cdots,-2+\frac34,\ \underbrace{-1+\frac34}_{-1-\frac14},\ \underbrace{1+\frac14}_{0-\frac14},\ 2+\frac14,\cdots\}$ $\{\cdots,-2+\frac14,-1+\frac14,1-\frac14,2-\frac14,\cdots\}$

$\cdots,\ \lambda_{-1},\ \lambda_0,\ \lambda_1,\ \lambda_2,\cdots$

(Why translation is allowed: $f \mapsto e^{-i\lambda t}f$ is $\boxed{\text{isomorphism}}$ in $L^2[-\pi,\pi]$, or $C[a,b]$.)

the translated λ_n satisfy $|\lambda_n| \le |n| + \frac{1}{2\times 2} = |n|+\frac14$, then by thm4 $\Rightarrow$ completeness!

To see the exactness, fails to be complete on the removal of any one term

Consider $f(z)=\int_{-\pi}^{\pi}[\cos(\frac t2)]^{-\frac12}e^{izt}\,dt$ bounded in $\mathbb{R}$ (not in PW)!

$$\int_{-\pi}^{\pi}(\cos\tfrac12 t)^{-1/2}e^{i\lambda_n t}\,dt = \sqrt2\int_{-\pi}^{\pi}(1+e^{it})^{-1/2}e^{int}\,dt$$
$$= \sqrt2\lim_{r\to 1-}\int_{-\pi}^{\pi}(1+re^{it})^{-1/2}e^{int}\,dt$$
$$= \sqrt2\lim_{r\to 1-}\sum_{k=0}^{\infty}\binom{-\frac12}{k}r^k\int_{-\pi}^{\pi}e^{i(n+k)t}\,dt = 0.$$

$\Rightarrow f(0)\neq 0,\ f(\lambda_n)=0,\ \forall n$

then $\forall n_0$, $\frac{f(z)}{z-\lambda_{n_0}} \in PW$, non-zero function. (remove 奇点)

$\Vert$ by PW

$\int_{-\pi}^{\pi}\phi(t)e^{izt}\,dt$ for some $\phi \in L^2\setminus\{0\}$, i.e. ϕ is orthogonal to $\{e^{i\lambda_n t}\}_{n\neq n_0}$

$\Downarrow$

$\{e^{i\lambda_n t}\}$ is exact by definition.

It remains to show that $\{e^{i\lambda_n t}\}$ is not a Riesz basis

Consider $f(z)=\frac{\Gamma^2(\frac34)}{\Gamma(\frac34+z)\cdot\Gamma(\frac34-z)}$, where $\Gamma^{-1}(z) = z\cdot e^{\gamma z}\prod_{n=1}^{\infty}(1+\frac zn)e^{-\frac zn}$ (Euler constant)

$\Downarrow$

$f(\lambda_n)=0$

and $f'(\lambda_n) = (-1)^n\cdot\Gamma^2(\frac34)\cdot\frac{\Gamma(n)}{\Gamma(n+\frac12)}$, $n>0$, and $\frac{\Gamma(n)}{\Gamma(n+\frac12)} \sim \frac{1}{\sqrt n}$ (omit the calculation)

$\boxed{\text{If } \{e^{i\lambda_n t}\} \text{ is a Riesz basis,}}$ then $1=\sum c_n e^{i\lambda_n t}$, $\boxed{\sum|c_n|^2<\infty}$ 之后用于Argue by contradiction.

then by taking Fourier transform

$\frac{\sin(\pi z)}{\pi z} = \sum c_n\cdot\underbrace{\frac{\sin(z-\lambda_n)}{\pi(z-\lambda_n)}}_{}$ $K_{\lambda_n}(z)$, reproducing kernel. , then

$\left\langle \frac{f(z)}{f'(\lambda_n)(z-\lambda_n)}, K_{\lambda_m}(z)\right\rangle = \delta_{n,m}$ $\Rightarrow$ $\{f_n\}$, $\{K_{\lambda_m}\}$ biorthogonal to each other

$\Vert$ def

f_n

$\Rightarrow \forall f$, $f=\sum(f,f_n)K_{\lambda_n}$ (by bi-orthogonality), then

$\frac{\sin(\pi z)}{\pi z} = \sum_n\underbrace{(\frac{\sin\pi z}{\pi z}, f_n)}_{\Vert}K_{\lambda_n}$

$c_n = f_n(0) = \frac{f(0)}{\lambda_n f'(\lambda_n)}$

$\Rightarrow |c_n| \sim \frac{1}{|\lambda_n|\cdot|f'(\lambda_n)|} \sim \frac{1}{\sqrt n}$ $\Rightarrow \sum_n|c_n|^2=\infty$. Contradiction! $\blacksquare$

Note：最早的关于 Riesz basis 问题：对 λ_n 扰动多少仍然为 Riesz basis

Kadec $\frac{1}{4}$-thm 基本上完全回答了这种问题，$\frac{1}{4}$ 也为 sharp 的.

Remark. By refining the argument, we can prove that the set

不单单不为 Riesz basis，甚至连 basis 都不是

$$\{e^{\pm i(n-1/4)t} : n = 1, 2, 3, \ldots\}$$

is not even a *basis* for $L^2[-\pi, \pi]$ (see Problem 1).

3.4: Some Intrinsic properties of sets of complex exponentials

observe $f \in PW$ (or $f \in L^2$), $f(\mu) = 0$, then $\frac{z-\lambda}{z-\mu} f(z) \in PW$. More generally, it holds in L^p

Thm 6. suppose $f(z) = \int_{-\pi}^{\pi} \alpha(t) e^{izt} dt$, $\alpha \in L^p[-\pi,\pi]$, $1 \le p < \infty$, $f(\mu) = 0$. and $g(z) = \frac{z-\lambda}{z-\mu} f(z)$ 换零点.

then $\exists \beta \in L^p[-\pi,\pi]$, s.t. $g(z) = \int_{-\pi}^{\pi} \beta(t) e^{izt} dt$. In fact

$$\beta(t) = \alpha(t) + i(\lambda - \mu) e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds.$$

proof: Step 1. $\frac{1}{z-\mu} f(z) = \frac{1}{z-\mu} \int_{-\pi}^{\pi} \alpha(t) e^{i\mu t} \cdot e^{i(z-\mu)t} dt$ → for integration by parts

$$= \frac{1}{z-\mu} \int_{-\pi}^{\pi} e^{i(z-\mu)t} d\left[\int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds\right]$$

$$= -i \int_{-\pi}^{\pi} \underbrace{\left(e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds\right)}_{\alpha_1(t)} e^{izt} dt$$

Step 2. $(z-\lambda) \int_{-\pi}^{\pi} \alpha_1(t) e^{izt} dt$

$$= (z-\lambda) \int_{-\pi}^{\pi} \boxed{\alpha_1(t) e^{i\lambda t}} \cdot e^{i(z-\lambda)t} dt$$

$$= \frac{1}{i} \int_{-\pi}^{\pi} (\alpha_1(t) e^{i\lambda t}) \, d e^{i(z-\lambda)t}$$

$$= -i \int_{-\pi}^{\pi} e^{izt} (\alpha_1'(t) + i\lambda \alpha_1(t)) dt$$

$$-i\mu e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds + e^{-i\mu t} \alpha(t) \cdot e^{i\mu t}$$

$$+ i\lambda e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds$$

$$= -i \int_{-\pi}^{\pi} e^{izt} \underbrace{\left[\alpha(t) + i(\lambda-\mu) e^{-i\mu t} \int_{-\pi}^{t} \alpha(s) e^{i\mu s} ds\right]}_{=\beta(t)} dt$$

□

Remark: It holds similarly on $C[a,b]$.

or consequence of the above thm

Thm 7: The completeness of $\{e^{i\lambda_n t}\}$ in $L^p[-\pi,\pi]$, $1 < p < \infty$, or $C[a,b]$ is unaffected if one of λ_n is replaced by another.

Thm 8: The system of $\{e^{i\lambda_n t}\}$ in $C[a,b]$ (L^p) is complete, if and only if $\operatorname{span}\{e^{i\lambda_n t}\}$ contains another exponent $e^{i\lambda t}$

Proof: "$\Rightarrow$" by completeness, trivial

"$\Leftarrow$" By translation, we may assume $\lambda = 0$, i.e. $1 \in \operatorname{span}$

We claim $f \in \operatorname{span} \Rightarrow \int_{-\pi}^{t} f \in \operatorname{span}$

If so $t, t^2, t^3, \cdots \in \text{span}$, as desired. $\forall \varepsilon > 0$, $\exists \sum_{n=1}^{N} c_n e^{i\lambda_n t}$, s.t. $\|f - \sum_{n=1}^{N} c_n e^{i\lambda_n t}\| < \varepsilon$

then $\left\| \int_{-\pi}^{t} f - \sum_{n=1}^{N} \frac{c_n}{i\lambda_n} e^{i\lambda_n t} - \sum_{n=1}^{N} \frac{c_n}{i\lambda_n} e^{-i\lambda_n t} \right\|$

$= \left\| \int_{-\pi}^{t} (f - \sum_{n=1}^{N} c_n e^{i\lambda_n t}) \right\| \leq \pi \cdot \|f - \sum_{n=1}^{N} c_n e^{i\lambda_n t}\| \leq \pi \cdot \varepsilon$. $\blacksquare$

corollary of thm 8

Thm 9: Every incomplete set of $\{e^{i\lambda_n t}\}$ must be minimal.

$\text{span}_{n \neq n_0} \{e^{i\lambda_n t}\} \neq \text{span}_n \{e^{i\lambda_n t}\}$, $\forall n_0$

proof: Corollary of thm 8. $\blacksquare$

前面thm的总结

Thm 10: $\{e^{i\lambda_n t}\}$ is either minimal or linked.

$\text{span}_{n \neq m} \{e^{i\lambda_n t}\} \neq \text{span}_n \{e^{i\lambda_n t}\}$ $\forall m$ $\qquad$ $e^{i\lambda_m t} \in \text{span}_{n \neq m} \{e^{i\lambda_n t}\}$, $\forall m$

proof: Incomplete $\xRightarrow{\text{thm 9}}$ minimal

complete, if minimal ✓, or assume $\{e^{i\lambda_n t}\}$ is complete but not minimal

$\exists n_0$ s.t. $\text{span}_{n \neq n_0} \{e^{i\lambda_n t}\} = \text{span}_n \{e^{i\lambda_n t}\}$

$\Rightarrow \{e^{i\lambda_n t}\}_{n \neq n_0}$ is complete.

by thm 7, $\{e^{i\lambda_n t}\}_{n \neq m}$, $\forall m \Rightarrow e^{i\lambda_m t} \in \text{span}_{n \neq m} \{e^{i\lambda_n t}\}$, $\forall m$ as desired. $\blacksquare$